ON THE VALIDITY RANGE OF PISTON THEORY

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Abstract: The basis of linear piston theory in unsteady potential flow is used in this work to develop a quantitative treatment of the validity range of piston theory. In the limit of steady flow, velocity perturbations from Donov's series expansion for the supersonic flow over an airfoil are used to assess the contributions of nonlinear terms relative to linear terms in the full perturbed potential equation. The range of Mach number and flow turning angle for which linear terms dominate is put forward as the analytical validity range for linear piston theory as based in potential flows. The range of validity of single-term nonlinear extensions to the linear potential equation into the transonic and hypersonic regions is treated. A brief review of the development of related aerodynamic methods for supersonic flows which share a similar form with piston theory is given; where applicable, the piston theory coefficients for the methods are given. The differences in the theoretical bases between the methods are highlighted. A discussion of the role of higher-order terms in piston theory and the validity limits of local piston theory is given.

1 INTRODUCTION

The simple formulation of piston theory in relating aerodynamic pressure and structural motion through a point-function relation has made it a popular analytical tool for aeroelasticians. The point-function relation is simple to implementation computationally, and thus piston theory has been used in several commercial tools for aeroelastic analysis, such as the ZAERO range of codes and Nastran. Piston theory has received renewed attention in recent literature [1–3], particularly in the use of local piston theory with CFD to reduce the computational cost of high-fidelity unsteady aerodynamic computations. The use of higher-order piston theory is widely used [4] in investigations of fluid-structure interactions of flexible panels and hypersonic vehicles. Lower-order piston theory shares a common form with other related aerodynamic methods, and the coefficients from methods with a different theoretical basis are often used in a piston theory formulation [5]. The aim of this work is two-fold: (1) to highlight the roots of piston theory and related aerodynamic methods in a common theoretical basis, and (2) to give an analytical treatment of the validity range of linear piston theory. This will serve as a basis for the treatment of extensions to nonlinear regions. A discussion of the role of higher-order terms in piston theory and of parameters influencing the validity of local piston theory will follow.

2 THEORETICAL BASIS

2.1 The piston analogy

The origins of piston theory lie in the analysis of hypersonic flows. The hypersonic equivalence principle is a concept which is central to the majority of aerodynamic methods for high supersonic speeds, and is the intuitive basis from which piston theory stems. The essence of the principle lies in the equivalence between steady flow in n spatial dimensions, and unsteady flow in $n-1$ dimensions; the reduction in the spatial dimension of the problem being compensated for through the exchange of one of the spatial variables for the time variable. This is essentially achieved through a Galilean transformation; the steady flow in an Euler reference frame then becomes unsteady flow in a plane which is normal to the direction of the Galilean transform. With this interpretation, the reduction in spatial dimension of unsteady flows is also possible.

The piston analogy lies in considering the cross-section of the body in the unsteady flow plane as a piston; the unsteady flow plane being the cylinder. The variation in the cross-section of the body in the direction of the Galilean transform is perceived as expansion or contraction (and in general, change of shape) of the piston. Rigid-body motion may then be perceived as additional translation of the piston within the plane. Furthermore, it may be seen that applying the Galilean transform in a direction which is not colinear with the freestream velocity vector effectively results in an additional steady "freestream" in the unsteady plane. The early use of the terms "piston" and "cylinder" in applying the unsteady analogy can be traced to Lighthill [6] and Hamaker et al [7].

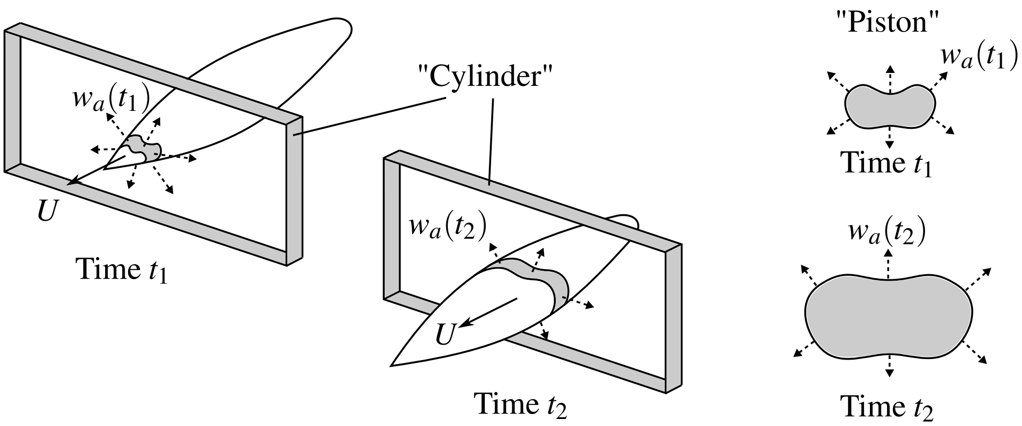


Fig. 1 The piston analogy in the hypersonic equivalence principle.

An important assumption made in the hypersonic equivalence principle is that the variation of the flow variables normal to the unsteady plane (i.e., in the direction of the Galilean transform) is negligible compared to variations in the unsteady plane. Hayes [8] formalized this assumption, and generalized the hypersonic equivalence principle of Tsien [9] to three-dimensional flows including shocks and flow rotationality, stating that in the limit of

$$M \gg 1, \qquad \theta \ll 1, \qquad K \gtrsim 1, \tag{1}$$

the steady flow in three-dimensions is equivalent to unsteady flow in a plane perpendicular to the steady motion of the body; here M is the freestream Mach number, θ is the local surface inclination to the undisturbed flow, and $K = M\theta$ is the classical hypersonic similarity parameter. The basis for this generalization was that disturbances at two points on the same streamline were in phase – Hayes' noted that this would be valid for large local Mach number of the flow; it may also be stated as a requirement that the Strouhal number must be small. Hayes thus laid the foundation for the piston analogy to be made for hypersonic flows. This was extended to three-dimensional unsteady flows by Hamaker et al [7] through potential flow analysis. Furthermore, Hamaker et al offered a review of research [10] into the validity range of the hypersonic similarity parameter $K = M\theta$. To estimate the validity range of the parameter, surface pressures on cones and ogives were calculated by the method of characteristics for a range of Mach numbers and body thickness ratios; the range of parameters for which the surface pressures could be collectively described to within a specified error bound by K was defined as the range of validity for the similarity law.

2.2 Lighthill's piston theory

Applying Hayes' equivalence principle, Lighthill [6] reasoned that in the absence of strong shocks, the pressure on the surface of an oscillating airfoil could be modelled, by physical analogy, using the pressure equation for a piston producing simple waves. In Lighthill's application, a number of assumptions were made, namely that the Mach number is sufficiently high (and shock and Mach angles are sufficiently small) that

1. gradients in flow quantities in the x-direction are "small" compared to gradients in the z-direction,
2. $\hat{u}$ is "small" compared to w,

where x is the direction of the undisturbed flow; z is perpendicular to x; $\hat{u}$ is the perturbation to the x-component of velocity, u; w is the z-component of velocity. In applying the simple wave pressure equation, Lighthill further assumed that the piston velocity remained subsonic. This was cast as

$$K\left[1+\left(\frac{z_0}{\theta c}\right)k\right] < 1, \tag{2}$$

$$k \equiv \frac{\omega c}{U}, \tag{3}$$

where c is the chord, z_0 is the amplitude of oscillation, U is the freestream velocity, ω is the angular frequency of oscillation, and k is the reduced frequency or Strouhal number. This leads from the equation for the downwash at the airfoil surface,

$$w_a = U f_x + f_t, \tag{4}$$

where w_a is the downwash, f is the equation describing the z-coordinate of the airfoil surface, and subscript notation denotes differentiation. Finally, the pressure equation for Lighthill's piston theory [6] is

$$\frac{p}{p_\infty} = \left[1+\frac{\gamma-1}{2}\left(\frac{w_a}{a_\infty}\right)\right]^{\frac{2\gamma}{\gamma-1}}, \tag{5}$$

where p is the pressure at the surface of the airfoil, p_∞ is the freestream pressure, and γ is the adiabatic exponent.

The general point-function relation for pressure for piston theory and similar theories is, up to third-order downwash terms,

$$\frac{p}{p_\infty} = 1+\gamma\left[c_1\left(\frac{w_a}{a_\infty}\right)+c_2\left(\frac{w_a}{a_\infty}\right)^2+c_3\left(\frac{w_a}{a_\infty}\right)^3\right], \tag{6}$$

for which expansion of Eq. (5) yields

$$c_1 = 1, \qquad c_2 = \frac{\gamma+1}{4}, \qquad c_3 = \frac{\gamma+1}{12}. \tag{7}$$

The classical review paper of Ashley and Zartarian [11] gives the validity limit of linear piston theory as any one of the following:

$$M^2 \gg 1, \qquad kM^2 \gg 1, \qquad k^2M^2 \gg 1. \tag{8}$$

2.3 Similar methods

2.3.1 Point-function relations in planar flows

The basis of Lighthill's [6] piston theory, as was seen in the previous section, lies in a physical argument based on Hayes' [8] hypersonic equivalence principle, with a pressure equation modelling simple waves. However, the general point-function relation for pressure of piston theory, as given by Eq. (6), is shared by other aerodynamic methods. The basis for these methods does not lie in Hayes' equivalence principle, and the methods are developed from general theory rather than by analogy. The works of Donov [12] and of Van Dyke [13] are considered, along with the relations for oblique shocks in the hypersonic limit. The methods listed were developed for steady flows, but may be expected to apply in the quasi-steady limit of $k \ll 1$.

Donov [12] developed a thorough treatment of the steady flow at the surface of a curved wall in supersonic flows through analysis by method of characteristics; the basis of his analysis is thus the Euler equations. The analysis assumes that shocks remain attached, and that the flowfield is everywhere supersonic. Whilst Donov presented detailed expressions for the flow velocity and pressure on the airfoil surface, accounting for the shock at the leading-edge and leading-edge curvature, a simplified form of the equations up to third-order in flow deflection θ is presented here. The truncated form of Donov's [12] expression for the flow velocity at the airfoil surface is

$$\frac{q}{U} = 1 + b_1\theta + b_2\theta^2 + b_3\theta^3 \qquad \text{for } \theta > 0, \tag{9}$$

$$\frac{q}{U} = 1 + b_1'\theta + b_2'\theta^2 + b_3'\theta^3 \qquad \text{for } \theta < 0, \tag{10}$$

where q is the local flow velocity, and $\theta > 0$ is associated with flow compression, and

$$b_1 = b_1' = -\frac{1}{m}, \tag{11a}$$

$$b_2 = b_2' = -\frac{1}{m^4}\left(\frac{1}{2} + \frac{\gamma-1}{4}M^4\right), \tag{11b}$$

$$b_3 = -\frac{1}{m^7}\left[\frac{1}{6} + \frac{1}{2}M^2 + \frac{3}{4}(\gamma-1)M^4 + \frac{3\gamma^2-12\gamma+5}{24}M^6 + \frac{(\gamma+1)^2}{32}M^8\right], \tag{11c}$$

$$b_3' = -\frac{1}{m^7}\left[\frac{1}{6} + \frac{1}{2}M^2 + \frac{3}{4}(\gamma-1)M^4 + \frac{2\gamma^2-5\gamma+3}{12}M^6\right], \tag{11d}$$

where $m^2 = M^2 - 1$. Donov's [12] expression for the pressure was given as a series in powers of the flow deflection, θ; for small perturbations such that $\sin\theta \approx \theta$ and $q \approx U$, it follows that

$$\frac{w}{a_\infty} = \frac{q}{a_\infty}\sin\theta \approx M\theta, \tag{12}$$

and hence Donov's pressure relation may be coaxed into the general form of Eq. (6). The corresponding piston theory coefficients for Donov's analysis are then

$$c_1 = \frac{M}{m}, \tag{13a}$$
$$c_2 = \frac{(\gamma+1)\,M^4 - 4m^2}{4m^4}, \tag{13b}$$
$$c_3 = \frac{1}{12Mm^7}\left[8 - 12M^2 + 9\,(\gamma+1)\,M^4 + \left(\frac{3\gamma^2 - 10\gamma - 4}{2}\right)M^6 + \left(\frac{3\gamma^2 + 6\gamma + 8}{8}\right)M^8\right] \tag{13c}$$
$$c_3' = \frac{1}{12Mm^7}\left[8 - 12M^2 + 10\,(\gamma+1)\,M^4 + \left(2\gamma^2 - 7\gamma - 5\right)M^6 + (\gamma+1)\,M^8\right], \tag{13d}$$

where c_3 is for compression, and c_3' is for expansion. It may be seen that up to second-order in flow deflection, the pressure relation is the same for compression and expansion; this is expected, as the effect of entropy from the shock is only introduced from third-order terms.

Van Dyke's [13] treatment of second-order supersonic flow is formulated from the basis of steady potential flow. Van Dyke's expression for the surface pressure on a curved wall may also be written in the form of Eq. (6) for small angles such that $\tan\theta \approx \theta$, and the corresponding piston theory coefficients from Van Dyke's second-order supersonic theory are

$$c_1 = \frac{M}{m}, \qquad c_2 = \frac{(\gamma+1)\,M^4 - 4m^2}{4m^4}. \tag{14}$$

It is seen that the coefficients agree with those given by Donov [12]. The potential flow result is in agreement with that of the method of characteristics up to the introduction of flow rotationality through terms of order θ^3.

Finally, the pressure behind an oblique shock for small wedge angles in the hypersonic limit (as is exploited in the tangent-wedge approximation and "strong shock piston theory") is given [14] by

$$\frac{p}{p_\infty} = 1 + \gamma K^2 \left[\frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}\,\right]. \tag{15}$$

This equation may be expanded in powers of K and be correlated to Eq. (6) to give piston theory coefficients as

$$c_1 = 1, \qquad c_2 = \frac{\gamma+1}{4}, \qquad c_3 = \frac{(\gamma+1)^2}{32}. \tag{16}$$

Comparison of the above coefficients to those obtained for expansion in the hypersonic limit, as given by Eq. (7), shows agreement up to second-order, with the difference due to rotationality in the third-order term noted.

2.3.2 Hypersonic small parameter expansions

Hypersonic small disturbance theories and expansions in small parameters are a class of methods which stand distinct from the methods considered thus far. The methods share commonality in the applicability of the hypersonic equivalence principle (unsteady analogy), but are different in the approach taken in the analysis. The previously mentioned works of Donov [12] and Van Dyke [13] are related, in that these works assume small disturbances to the flow parameters. In the methods that are to be discussed in this section, the smallness of some parameter in the flow (or of a geometric ratio of the body) is exploited in reducing the complexity of, typically, the Euler equations.

An early development of hypersonic small disturbance theory contemporary of Van Dyke [14] was that of Il'yushin [15]. Il'yushin offered a rigorous development of the hypersonic equivalence principle, which was named the law of plane sections; as noted by Hayes and Probstein [16], it was only published in accessible literature in 1956, despite being authored earlier. Furthermore, a hypersonic small disturbance theory was developed [15], and subsequently tied to linearized aerodynamic theory and to piston theory. In Il'yushin's work, the Euler equations were used in treating the flow over a slender body with assumptions similar to Lighthill's, but more rigorous, in that

$$M \gg 1, \qquad w \approx \theta u, \qquad \theta \ll 1. \tag{17}$$

Following these assumptions, Il'yushin [15] showed that up an accuracy of $\theta^2 + 1/M^2$, the mass of fluid between two planes perpendicular to the x-direction remains unchanged as it passes over the body, and that the pressure is dependent only on motion of the fluid in planes perpendicular to the x-direction. The relative smallness of order $\theta + 1/M$ of pressure gradients and of order θ of velocity gradients in the x-direction compared to gradients in the z-direction follow from his analysis. The applicability of strip theory for three-dimensional bodies was also treated. In formulating the equations for the surface pressure, Il'yushin distinguished between expansion and compression of the flow, resulting in the hypersonic forms of the pressure equation for Prandtl-Meyer expansion and oblique shocks, respectively (although up to the second-order in θ, these are equivalent). As such, Il'yushin gave a rigorous framework for the derivation of the piston theory pressure equations at hypersonic speeds from the basis of the Euler equations.

A further significant development to hypersonic small disturbance theory was the formulation for slender bodies at high angles of attack due to Sychev [17]. Sychev noted that with increasing angle of attack, flow perturbations are no longer small; however, Sychev showed that a useful simplification of the equations of motion is achieved if the thickness and aspect ratio of the body is considered a small parameter. Sychev further assumed that the crossflow Mach number is hypersonic, and thereby neglected the influence of the lee-side flow. The following assumptions were made:

$$\delta \ll 1, \qquad M\delta \gtrsim 1, \qquad M \sin\alpha \gg 1, \qquad \alpha > \delta. \tag{18}$$

Here, δ represents the ratio of the largest transverse dimension of the body to the chord length, and α is the angle of attack. After simplification through neglecting terms of second-order and higher in δ, Sychev [17] obtained the large-incidence formulation for hypersonic flow, and noted that the flow solution depends on two parameters: $K_1 = \delta \cot\alpha$ and $K_2 = M \sin\alpha$; for small angles of attack such that $\alpha \sim \delta$, the large-incidence formulation agrees with that of small-disturbance theory (cast as $K = M\delta$). The formulation was developed from the Euler equations and shock relations, and Sychev applied the unsteady analogy (equivalence principle, law of plane sections) in planes perpendicular to the body axis; this formulation resulted in the piston having a velocity of $U \sin\alpha$ in addition to velocity due to rigid body motion and local surface deformations.

Barnwell [18] expanded on Sychev's [17] work, noting that the Sychev parameter $K_1 = \delta \cot \alpha$ appears in the equations of motion without any assumptions regarding Mach number and without the use of shock relations. Barnwell showed that the second Sychev parameter $K_2 = M \sin \alpha$ enters through the shock boundary conditions, and argued that Sychev's similarity parameters are in fact valid for any supersonic, slender-body flow, provided that the crossflow Mach number is supersonic ($M \sin \alpha > 1$). Barnwell further remarked that Sychev's formulation holds for subsonic crossflow Mach numbers provided that the flow is conventionally hypersonic; Barnwell's [18] extension of Sychev's conditions is then

$$\delta \ll 1, \qquad M \gg 1, \qquad M\delta \gtrsim 1, \qquad \text{for} \quad M \sin \alpha < 1, \tag{19}$$

$$\delta \ll 1, \qquad \text{for} \quad M \sin \alpha > 1. \tag{20}$$

The successful correlation of a range of experimental data for sharp-edged delta wings and slender bodies using Sychev's similarity parameters by Hemsch [19] gave empirical evidence that the validity of Sychev's parameters extends well into the subsonic crossflow range (in certain cases down to $M \sin \alpha = 0.2$). Barnwell [18] also briefly considered a linearized potential flow formulation for bodies at angle of attack, and noted that the resulting equation cannot be expressed simply in terms of the Sychev similarity parameters. Reduction of the number of parameters in the equation is only achieved for $M \sin \alpha \ll 1$, resulting in the classical linear potential flow equation.

The extensions to Sychev's [17] theory for slender bodies at large angles of attack were summarised by Voevodenko and Panteleev [20], who investigated the validity of and error in applying the law of plane sections (the hypersonic equivalence principle) to delta wings of varying aspect ratio and angle of attack. For delta wings with shocks attached to the windward edges, the error in calculated surface pressures in applying the law of plane section relative to an exact shock solution was found [20] to increase with increasing leading-edge sweep; it was noted that this was due to the decrease in the Mach number of the flow normal to the leading edge. Significantly, Voevodenko and Panteleev estimate the validity range in applying the law of plane sections in terms of angle of attack and leading-edge sweep angle; flows with shocks attached or detached from the wing leading-edges are considered, and it was noted [20] that for low leading-edge sweep angles, the dependence of the solution of the spanwise coordinate diminishes, and the transition to strip theory occurs (effectively further reducing the dimension of the unsteady analogy to a one-dimensional piston).

2.4 Extensions to piston theory

Returning to consideration of flows of more general Mach number, it has previously been shown [21] that in the for highly unsteady flows, linearized unsteady potential flow yields the same pressure relation as linear classical piston theory; in the quasi-steady limit, linearized supersonic theory is obtained (which may also be described in a piston theory formulation). The linearization of the equation for potential flow and of the boundary conditions allows for simplifcation of the analytical formulation; in particular, it allows the differential equation to be solved in the frequency domain. The planar linearized potential equation for simple harmonic motion in the time-domain may be written [21] as

$$\left(1 - M^2\right) \bar{\phi}_{xx} + \bar{\phi}_{zz} = -\frac{\omega^2}{a_\infty^2} \bar{\phi} + 2iM \frac{\omega}{a_\infty} \bar{\phi}_x, \tag{21}$$

where $\bar{\phi}$ is the amplitude of the harmonic potential function, and a_∞ is the speed of sound in the freestream. The Laplace transform of the above equation is written as

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} = \mu^2\Phi, \tag{22}$$

$$\mu^2 \equiv \left(M^2 - 1\right)\xi^2 + 2iM\frac{\omega}{a_\infty}\xi - \frac{\omega^2}{a_\infty^2}, \tag{23}$$

where Φ is the Laplace transform of the harmonic potential function, ξ is the Laplace transform of the streamwise coordinate (x), and μ represents elements of the potential equation associated with streamwise derivatives and unsteady terms. The solution for the potential function in the time domain is then given [21] by

$$\bar{\phi}(x, z = 0) = -\int_0^x \bar{w}_a(\chi)\mathscr{L}^{-1}\left\{\frac{1}{\mu}\right\}d\chi, \tag{24}$$

where $\bar{w}_a$ is the downwash at the airfoil surface, χ is a dummy integration variable for x, and $\mathscr{L}^{-1}$ represents the inverse Laplace transform. The inversion of the solution back to the time domain results in the introduction of non-trivial integrals [21], and the physical interpretation of the result becomes obscured. Dowell and Bliss [22] recently revisited the linearized potential flow treatment and considered the result of expanding the $1/\mu$ term in powers of ξ prior to inversion. The resulting power series in terms of ξ yields [22] a series of successive streamwise integrals upon inversion to the time domain. The first terms in the series represent classical linearized piston theory, with successive terms representing corrections which improve the piston theory approximation of the potential flow solution through accounting for upstream influence in the unsteady terms; as is noted by Dowell and Bliss, classical piston theory results from neglecting these upstream influences. The basis for the "extended piston theory" remains linearized potential flow.

The assumption of small flow perturbations, which is inherent in the linearization of potential flows, was exploited in creating what has become known as local piston theory (LPT). The essence of local piston theory is the use of linear piston theory to model unsteady (or steady) perturbations about an existing solution of the mean steady flow. The chief merit of LPT lies in the characteristic point-function relation between surface pressures and perturbations to the surface geometry; it allows for computationally inexpensive modelling of perturbed pressures to an existing flow. Furthermore, it allows for the modular use of other, more accurate aerodynamic methods in the computation of the steady flow, which typically dominates the solution of flows with small reduced frequencies. Works representative of developments in LPT include those of: Yates and Bennett [23], who investigated supersonic and hypersonic flutter using LPT to model unsteady perturbations about a shock-expansion solution; Ericsson [24], who considered viscous perturbations to inviscid flows, and modelled the effect of change in the slope of the inviscid flow due to the boundary layer using a first-order LPT over a tangent-wedge solution – this laid the foundations for a "viscous local piston theory"; Zhang et al [1], who used LPT to model unsteady perturbations about a mean steady Euler solution, and achieved excellent agreement to fully unsteady Euler computations for reduced computational cost; Han et al [25], who extend on the approach of Zhang through providing a viscous correction to the LPT contribution.

In closing, it is noted that in applying local piston theory, its basis as a perturbation to a mean flow must be considered; the perturbation is relative to the local "cylinder" conditions, with the "cylinder" oriented normal to the mean local flow. This is elaborated on in [26].

3 VALIDITY RANGE

3.1 Objectives

The relation of piston theory to linearized unsteady potential flow was noted in the previous section; particular attention is drawn to the fact [21] that the general formulation in terms of arbitrary reduced frequency k is equivalent to linear piston theory in both the quasi-steady limit ($k \to 0$) and in the limit of highly unsteady flow ($k \to \infty$). Dowell and Bliss [22] note that the classical piston theory limit results as the streamwise dependence is neglected ($\xi \to 0$ in the frequency domain). The frequency-domain analysis from which the "extended piston theory" of Dowell and Bliss [22] stems is dependent on the linearization of the unsteady potential equation. Similarly, local piston theory depends on this linearization.

It is thus of merit to consider the conditions under which linearization of the equation for unsteady potential flow becomes invalid. The validity range for piston theory, as based in a potential flow formulation, may then be quantitatively estimated as being the portion of the parameter space for which the contribution of nonlinear terms in the full perturbed unsteady potential equation is negligible. It should be noted that the aim of the present work is not to explore the validity of the law of plane sections (hypersonic equivalence principle); rather, in the spirit of the categorization of the applicability of particular methods to particular flow regimes made by Voevodenko and Panteleev [20], the aim is to establish a range of applicability for the linearized potential-flow formulation of piston theory. The validity of other related methods which may be cast in a piston theory formulation is not investigated.

The analytical development which follows is based on the use of Donov's [12] series for the velocity at the surface of a smoothly curved wall to correlate the magnitude of velocity perturbations with Mach number and flow turning angle. The contributions of nonlinear perturbations in the potential equation relative to linear perturbations may then be estimated. Nonlinear contributions are considered negligible if they are smaller than a specified multiple of the linear contribution; the variation of the validity range of the linearization with the smallness of the chosen factor may then be explored. The use of Donov's series limits the study to steady flows; thus, the conclusions are limited to cases in which the reduced frequency of the flow is very small. Furthermore, in keeping with the potential flow analysis and small perturbations basis of Donov's [12] expansion, it is assumed that the flow is everywhere supersonic, and that the wall curvature is small; the small curvature of the wall is consistent with the assumption of low reduced frequency in the unsteady analogy. Finally, it is assumed that the flow is turned smoothly; that is to say, the body is free of corners, and the flow behind the leading-edge shock or expansion-wave is considered.

3.2 Analytical formulation

The full pertubed potential equation for steady planar flows may be written as

$$(L_x - N_x)\,\hat{\phi}_{xx} + (L_z - N_z)\,\hat{\phi}_{zz} - B = 0, \tag{25}$$

where the following groupings of terms have been made: linear terms are grouped as

$$L_x = a_\infty^2 - U^2, \tag{26a}$$

$$L_z = a_\infty^2, \tag{26b}$$

and nonlinear terms are grouped as

$$N_x = X_1/e + X_2/e + Z, \tag{27a}$$
$$N_z = X_1 + X_2 + Z/e, \tag{27b}$$
$$B = 2\left(U + \hat{\phi}_x\right)\hat{\phi}_z\hat{\phi}_{xz}, \tag{27c}$$

with the symbols denoting the following terms:

$$e = \frac{\gamma - 1}{\gamma + 1}, \tag{28a}$$
$$X_1 = (\gamma - 1)\,U\hat{\phi}_x, \tag{28b}$$
$$X_2 = (\gamma - 1)\,\hat{\phi}_x{}^2/2, \tag{28c}$$
$$Z = (\gamma - 1)\,\hat{\phi}_z{}^2/2. \tag{28d}$$

In the preceding equations, L_i represents terms in linearized pertubation theory, N_i represents the nonlinear terms, B represents terms relating to flow curvature, and e is the density ratio across a shock in the hypersonic limit [16]; terms of the form $\hat{\phi}_i$ represent perturbation velocities. In the following analysis, it is assumed that B may be discarded, assuming small flow curvature.

In relating Donov's [12] expansion for velocity to pertubation velocities, the following preliminaries are noted:

$$u = U + \hat{\phi}_x = q\cos\theta, \tag{29}$$
$$w = \hat{\phi}_z = q\sin\theta, \tag{30}$$
$$\cos\theta \approx 1 - \theta^2/2, \tag{31}$$
$$\sin\theta \approx \theta - \theta^3/6. \tag{32}$$

Hence, up to third-order in flow turning angle, the perturbation velocities are estimated from Donov's series as:

$$\hat{\phi}_x/U = b_1\theta + (b_2 - 1/2)\,\theta^2 + (b_3^* - b_2/2)\,\theta^3 + O(\theta^4), \tag{33}$$
$$\hat{\phi}_z/U = \theta \quad + b_1\theta^2 \qquad\qquad + (b_2 - 1/6)\,\theta^3 \; + O(\theta^4), \tag{34}$$

in which b_3^* refers to b_3 or b_3' depending on the direction of flow turning, and where the coefficients are defined as in Eq. (11); the notation $O(\theta^4)$ denotes terms of fourth-order in flow turning angle. The following expressions are obtained for the nonlinear contributions in the potential flow equation:

$$X_1/U^2(\gamma - 1) = b_1\theta + (b_2 - 1/2)\,\theta^2 + (b_3^* - b_2/2)\,\theta^3 \; + O(\theta^4), \tag{35}$$
$$X_2/U^2(\gamma - 1) = \qquad \left(b_1^2/2\right)\theta^2 \quad + b_1(b_2 - 1/2)\,\theta^3 + O(\theta^4), \tag{36}$$
$$Z/U^2(\gamma - 1) = \qquad (1/2)\,\theta^2 \quad + b_1\theta^3 \qquad\qquad + O(\theta^4). \tag{37}$$

A preliminary observation may be made regarding the boundedness of these terms. When considering the asymptotic behaviour of the coefficients given by Eq. (11), it is seen that as $M \to \infty$, the second-order terms $b_1, b_2 \to 0$ and for expansion $b_3' \to 0$. However, the third-order term for compression flows becomes unbounded as $b_3 \sim (\gamma + 1)^2 M/32$. Thus, in the hypersonic limit of $M \to \infty$, the nonlinear contributions of Eqs. (36, 37) up to and including $O(\theta^3)$ remain bounded; the boundedness of the nonlinear contribution of Eq. (35) is determined at $O(\theta^3)$ by whether the flow is compressive or expansive.

Having thus obtained expressions for the nonlinear terms in the potential equation as a function of M and θ, what remains the treatment of negligibility. The mathematical notion of the symbol $\ll$ in asymptotics, whilst powerful in gaining insight to the physical problem, does not offer much help in quantifying when a parameter is "small enough." Similarly, the big-O notation of $O(1)$ strictly relates to boundedness, rather than to a sense of "order of magnitude."

The treatment of negligibility which is adopted in this work is intended as a framework for quantitatively assessing the relative importance on terms, and for quantifying the adjective "small." If the parameter ε is introduced as a measure of smallness, then negligibility may be defined by

$$X_1 \ll X_2, \quad \text{if} \quad \left|\frac{X_1}{X_2}\right| < \varepsilon. \tag{38}$$

Some insight may be gained into the degree of nonlinearity in the flow through variation of the smallness parameter ε.

3.3 Relative contributions of terms

The nonlinear contributions N_x and N_z, given by Eqs. (27a, 27b), to the full perturbed potential equation are functions of the perturbation velocities through Eqs. (28b – 28d). The variation of the dimensionless perturbation velocities with Mach number for a flow turning angle of 15° is given in Figure 2.

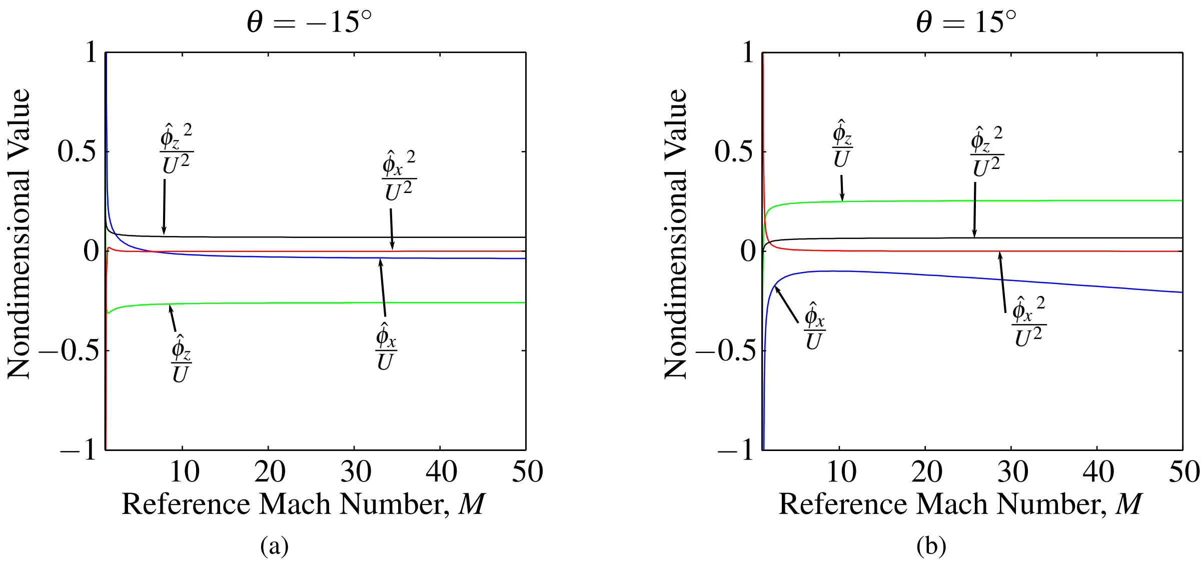


Fig. 2 Variation of dimensionless perturbation velocities with Mach number for: (a) expansion (b) compression.

As was noted previously regarding the boundedness of Eqs. (35 – 37), the dimensionless velocities remain bounded in the limit of $M \to \infty$, with the exception of $\hat{\phi}_x/U$ in compression flows. A further observation from Figure 2 is that for compression flows, $\hat{\phi}_x$ and $\hat{\phi}_y$ are of opposite sign for all (supersonic) Mach numbers for which the flow remains attached; in the case of expansion flows, however, the sign of $\hat{\phi}_x$ changes at some intermediate Mach number, above which it is of the same sign as $\hat{\phi}_y$. This behaviour is a function of the flow turning angle, and leads to the possibility of discontinuous ranges of M and θ over which the contribution may be neglected.

In considering the contribution of the nonlinear terms relative to linear terms in Eq. (25), the distinction must be made between the coefficients of $\hat{\phi}_{xx}$ and of $\hat{\phi}_{zz}$, as differences enter through the scaling of the density ratio e and through differences in L_x and L_z. The contributions of the nonlinear terms in the coefficient of $\hat{\phi}_{xx}$ are shown in Figure 3 for a flow turning angle of 15°; the coefficient of $\hat{\phi}_{zz}$ is shown in Figure 4

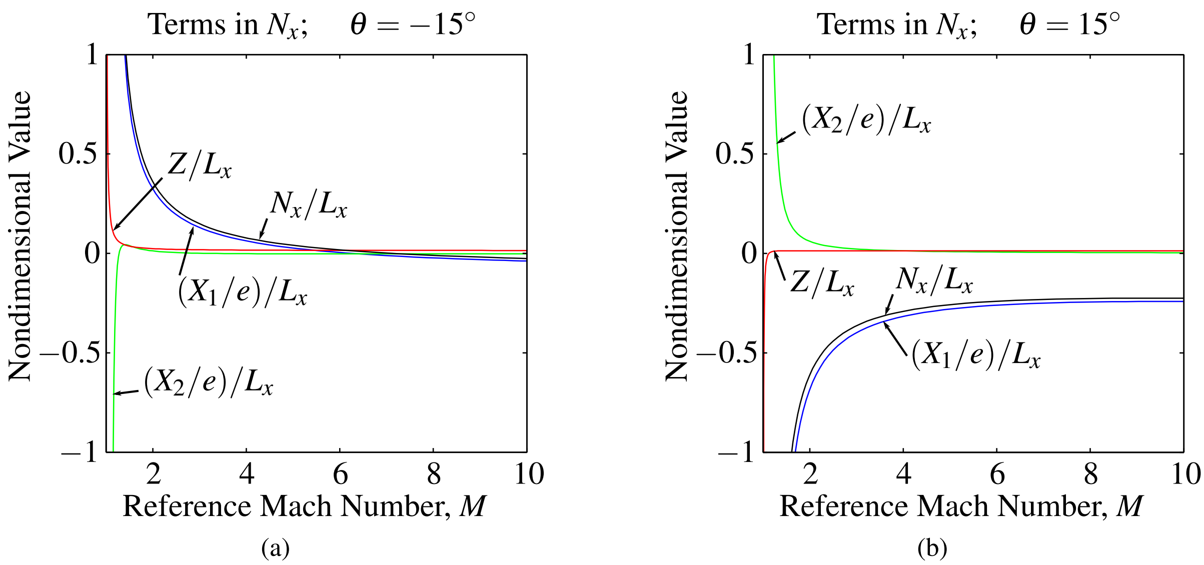


Fig. 3 Relative contribution of nonlinear terms in N_x for: (a) expansion (b) compression.

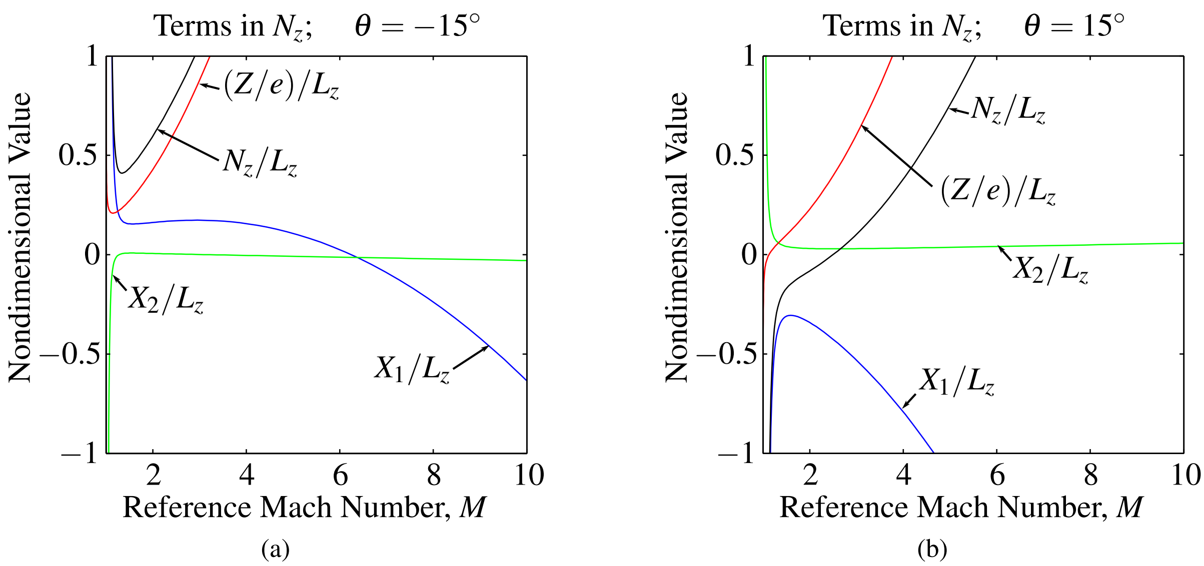


Fig. 4 Relative contribution of nonlinear terms in N_z for: (a) expansion (b) compression.

Regarding nonlinearity in $\hat{\phi}_{xx}$, Figure 3 shows that for both expansion and compression flows, the nonlinearity is dominated by the contribution from the term (X_1/e). Furthermore, it is noted that over the Mach range shown, the nonlinearity dominates in the low supersonic range; for compressive flows, the nonlinearity will also grow in the hypersonic limit, as noted from previous remarks regarding the boundedness of X_1/U^2.

The nonlinearity in $\hat{\phi}_{zz}$, as shown in Figure 4, exhibits different behaviour. As previously noted, the sign of the contribution from X_1 changes in expansion flows; however, for the depicted range of Mach numbers for this turning angle, the nonlinearity in expansion flows is dominated by the contribution from Z/e. The nonlinearity is significant, and grows with increasing Mach number from the transonic limit. In considering expansion flows, the contributions of X_1 and Z/e are seen to always be of opposite sign; in addition, there is a range of Mach numbers over which the contributions are of similar magnitude, and therefore have no net contribution. For compressive flows, the nonlinearity in the coefficient of $\hat{\phi}_{zz}$ is thus nontrivial.

3.4 Validity range of linearization

From consideration of the variation in nonlinearity in the coefficients of $\hat{\phi}_{xx}$ and $\hat{\phi}_{zz}$ with M and θ, the range of validity of linearization may be established as those values of M and θ for which the relative nonlinear contribution is smaller than a parameter ε. It should be noted that the overall contribution of the nonlinear terms is considered; in the framework adopted for considering the smallness of the contributions, this is an important distinction. As observed for expansion flows in Figure 4(a), individual contributions may be of similar magnitude, but opposite sign; thus, whilst the contribution of the individual terms may not be negligible, the net contribution of the terms will be. In the case of contributions having the same sign, neglecting the net contribution becomes a stronger limiting condition than the limits on individual terms; in the case of terms with opposite signs, the limiting condition in neglecting the net contribution poses a relaxed restriction on smallness of the individual terms.

The linearization of the potential equations is defined by the following conditions:

$$N_x \ll L_x, \quad \text{if} \quad \left|\frac{N_x}{L_x}\right| < \varepsilon, \tag{39a}$$

$$N_z \ll L_z, \quad \text{if} \quad \left|\frac{N_z}{L_z}\right| < \varepsilon, \tag{39b}$$

leading to the linearized potential equation in the form of

$$L_x\hat{\phi}_{xx} + L_z\hat{\phi}_{zz} = 0. \tag{40}$$

Linearization requires that Eqs. (39a, 39b) be simultaneously satisfied. Each equation has contribution from three terms, leading to six individual conditions for smallness of parameters which must be met. These equations may be considered separately to determine if the nonlinearity is dominated by an individual contribution, in order that the number of conditions to be met may be reduced to the strongest restrictions.

To consider satisfying Eq. (39a), the validity range of neglecting individual contributions to N_x, along with the validity of neglecting N_x collectively, is shown in Figure 5 for a smallness parameter ε of 0.10. For both expansion and compressive flows, it is evident that for the range of M and θ considered, the nonlinearity N_x is dominated by the contribution from X_1/e, as was noted in the discussion of Figure 3; once again it is seen that the nonlinearity for small flow turning angles is significant at low supersonic Mach numbers. The dominant condition to satisfy Eq. (39a) is thus $(X_1/e) \ll L_x$. It is also of significance that for flow compression, there is a moderate limiting flow turning angle above which the flow is nonlinear for all supersonic Mach numbers. In flow expansion, it is seen that for flow turning angles greater than 20°, the band of Mach numbers for which the flow may be linearized diminishes rapidly.

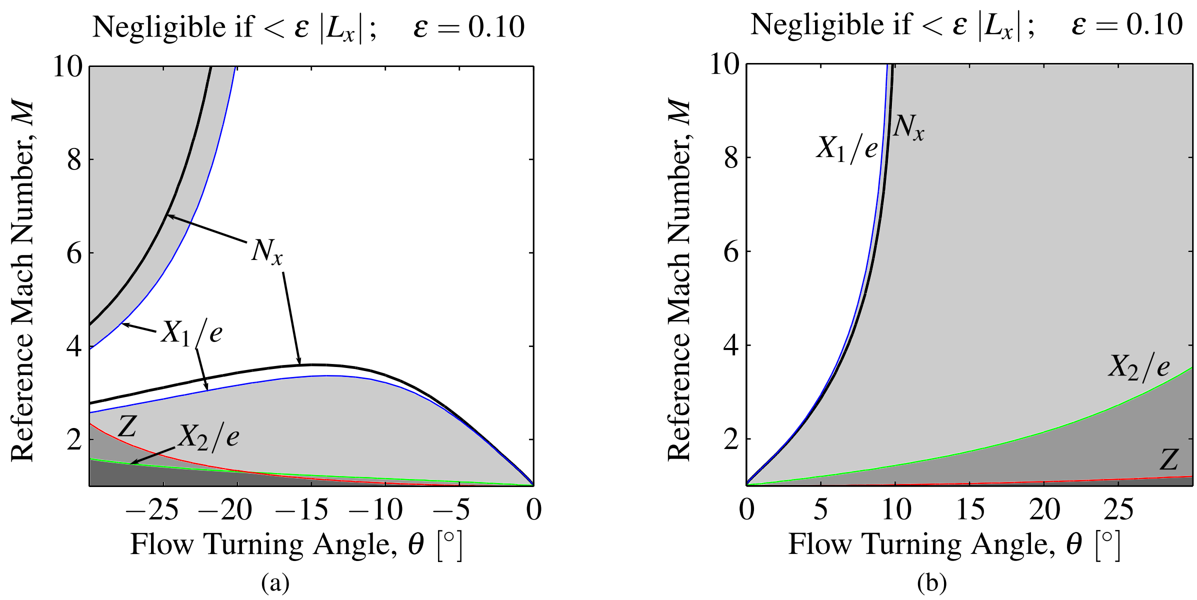


Fig. 5 Validity of neglecting contributions to the coefficient of $\hat{\phi}_{xx}$ for: (a) expansion (b) compression.

The regions in which nonlinear terms in the coefficient of $\hat{\phi}_{zz}$ become significant up to ε of 0.10 are shown in Figure 6. The greater complexity of the nonlinearity is immediately evident in the disparity between the individual ranges of negligibility of X_1 and Z/e and the range of validity in neglecting the net contribution N_z.

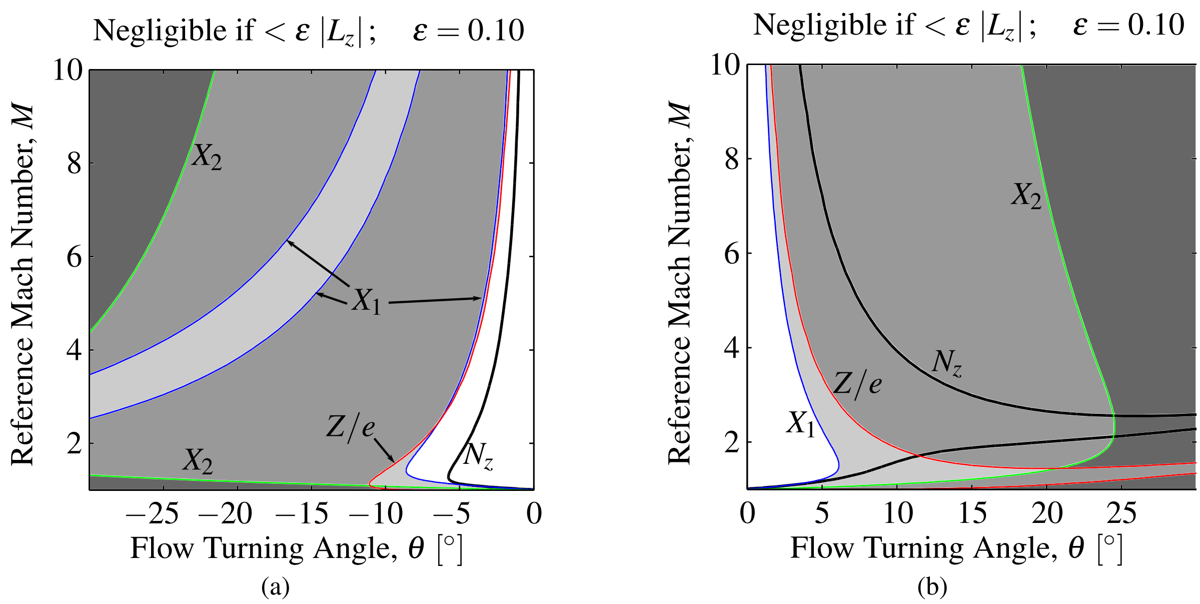


Fig. 6 Validity of neglecting contributions to the coefficient of $\hat{\phi}_{zz}$ for: (a) expansion (b) compression.

In considering flow expansion, the discontinuity in the range of negligibility of X_1 that was explained in the discussion of Figure 4 is observed. The insignificance of the contribution by X_2 is evident; in determining the dominant contribution, Figure 4(a) provides more insight. It may be expected that for the specified smallness parameter of $\varepsilon = 0.10$, the nonlinearity is mostly dominated by Z/e over the Mach range in Figure 6 for $\theta > 10^\circ$ from $M > 2$. However, the generalization is dependent on the value of ε chosen. In conclusion, it is noted that for expansion flows, the restriction of Eq. (39b) is not readily simplified.

Consideration of the linearization of the coefficient of $\hat{\phi}_{zz}$ for compression flows in Figure 6(b) shows that restrictions on the smallness individual contributions to the nonlinearity N_z are in fact too severe. Once again, for the portion of the parameter space considered, the contribution from Z/e is somewhat larger than that from X_1 (as per Figure 4(b)), and it is concluded that Eq. (39b) is not readily simplified.

Having considered Eqs. (39a, 39b) separately, the validity range of linearization by requiring simultaneous satisfaction of both equations is now shown for various smallness parameters ε in Figure 7.

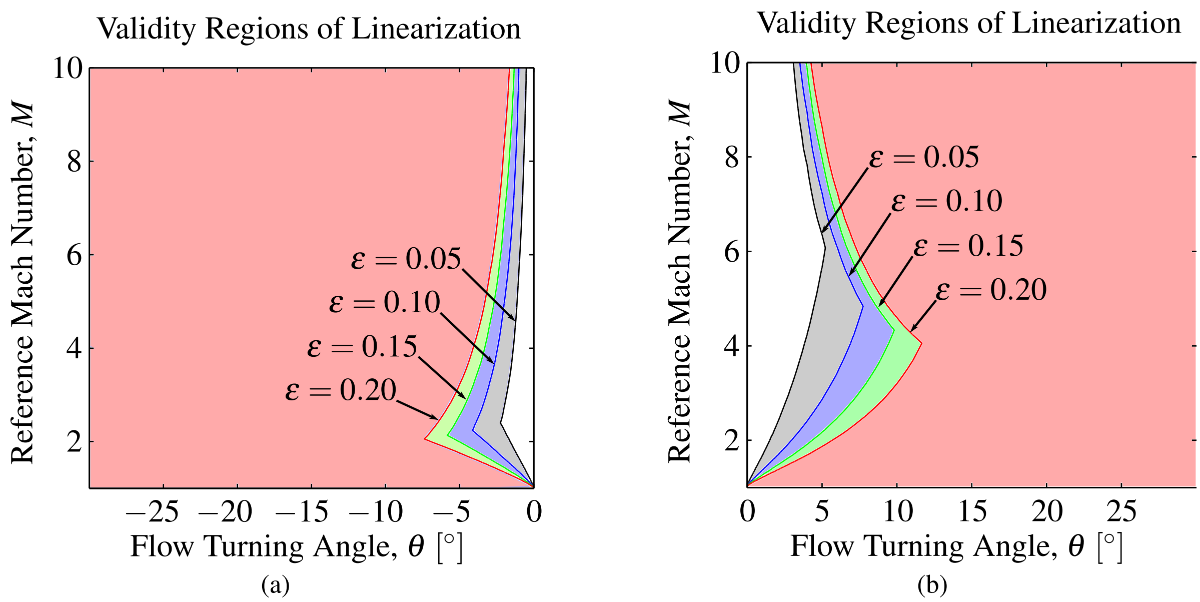


Fig. 7 Range of validity of linearization of the potential flow equation for: (a) expansion (b) compression.

Referring of Figure 5 and Figure 6, it is seen that for both expansion and compression flows, the restriction on linearization at higher Mach numbers is from the coefficient of $\hat{\phi}_{zz}$ via Eq. (39b); the restriction in the transonic limit is from the coefficient of $\hat{\phi}_{xx}$ via Eq. (39a), or essentially via $(X_1/e) \ll L_x$. The growth in the range of validity of linearized potential flow with relaxation of the severity of the smallness parameter ε is evident.

3.5 Extension to nonlinear regions

The previous consideration of individual terms dominating the nonlinearity in the discussions of Figure 5 and Figure 6 allows for extensions to be made to nonlinear regions through the addition of single terms. Whilst the nonlinearity violates the frequency-domain analysis basis of piston theory, it provides insight into the nonlinearity which may be used to guide the development of more detailed approximate models of the flow.

As previously noted, in the transonic limit, nonlinearity is dominant in the coefficient of $\hat{\phi}_{xx}$, and this nonlinearity is dominated by the contribution of X_1/e. If this contribution in not neglected in favour of the linear term L_x, then a transonic approximation for potential flow may be made as

$$L_x\hat{\phi}_{xx} + L_z\hat{\phi}_{zz} = (X_1/e)\hat{\phi}_{xx}. \tag{41}$$

subject to the conditions

$$X_2/e + Z \ll L_x - X_1/e, \quad \text{if} \quad \left|\frac{X_2/e+Z}{L_x - X_1/e}\right| < \varepsilon, \tag{42}$$

$$N_z \ll L_z, \quad \text{if} \quad \left|\frac{N_z}{L_z}\right| < \varepsilon. \tag{43}$$

The resulting extension to the transonic region is shown in Figure 8, and the validity range of the transonic approximation is shown in Figure 9.

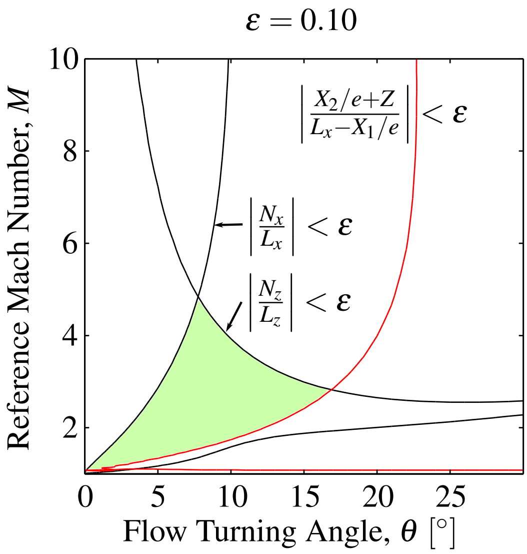
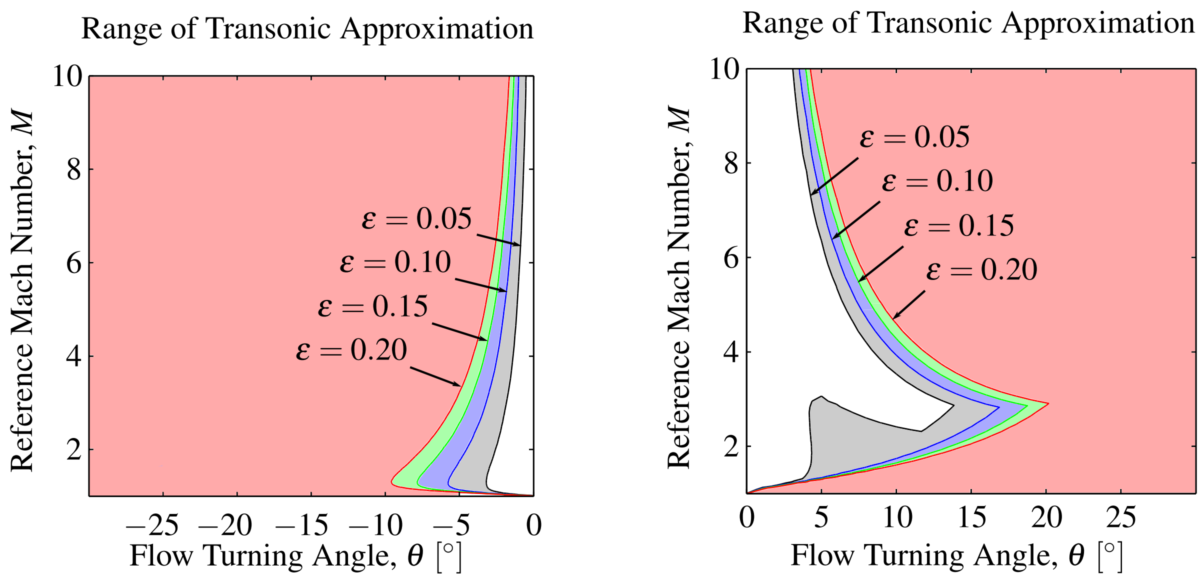


Fig. 8 Extension to transonic region through X_1/e in the coefficient of $\hat{\phi}_{xx}$.



(a) (b)

Fig. 9 Range of validity of the transonic approximation to the potential flow equation.

A similar extension to the hypersonic range may be attempted, but as remarked in the discussion of Figure 4, the nonlinearity N_z in the coefficient of $\hat{\phi}_{zz}$ is nontrivial due to the similar magnitudes of X_1 and Z/e. Whilst the validity region will not be explored in this work, an outline of a hypersonic approximation involving a single term will be given. If the ratio between the dominant nonlinear is taken as r, where

$$r = \frac{X_1}{Z/e}, \tag{44}$$

and r is taken to be constant, then a hypersonic approximation for potential flow may be made as

$$L_x \hat{\phi}_{xx} + L_z \hat{\phi}_{zz} = (rZ/e)\hat{\phi}_{zz}. \tag{45}$$

subject to the conditions

$$N_x \ll L_x, \qquad \text{if} \quad \left|\frac{N_x}{L_x}\right| < \varepsilon, \tag{46}$$

$$X_2 \ll L_y - rZ/e, \quad \text{if} \quad \left|\frac{X_2}{L_y - rZ/e}\right| < \varepsilon. \tag{47}$$

The chief difficulty in applying this approximation lies in the variability of r with M and θ, which is shown in Figure 10. The range of Mach numbers over which r may be approximated to accuracy of ε as a constant is seen to be greatest for large turning angles in expansion flows, and for small turning angles in compression flows; the latter is consistent with classical assumptions of small incidence in hypersonic flows.

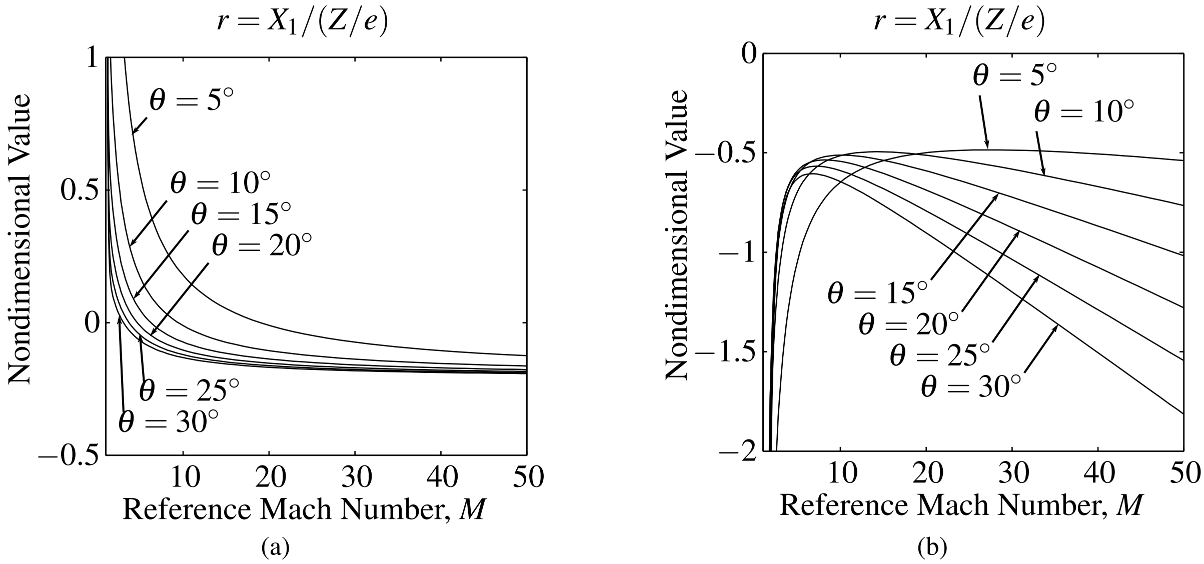


Fig. 10 Ratio of dominant nonlinear terms in the coefficient of $\hat{\phi}_{zz}$ for: (a) expansion (b) compression.

4 DISCUSSION

4.1 Relation of analysis to previous work

The preceding analytical treatment of the validity of linearization for steady potential flow serves to provide a quantitave estimate of the range of Mach number and flow deflection angle in which linear piston theory – as stemming from linearized potential flow – may be applied in the quasi-steady limit. The survey by Hamaker et al [7] in studying the accuracy of Hayes' hypersonic parameter ($K = M\theta$) in correlating surface pressures on bodies of revolution from method of characteristics solutions represents an early attempt at establishing a validity range for piston theory (as rooted in Hayes' hypersonic equivalence principle). The work of Voevodenko and Panteleev [20] represents an approximate categorization of the parameter space for delta-wing flows at hypersonic Mach numbers into regions across which various hypersonic aerodynamic methods apply. The present work may be viewed partially as an extension of the categorization of Voevodenko and Panteleev [20] to include potential flows, and in adding an extra parameter in the form of Mach number variation; the work is also similar in it's approach to that of Hamaker et al [7], but provides deeper insight into the validity of the linearization, the range of validity, and the sources of nonlinearity. The potential flow treatment, by extension, provides a range of validity for the extended piston theory of Dowell and Bliss [22] at low reduced frequencies.

4.2 Higher order terms

The contribution of terms nonlinear in θ may be correlated to nonlinearities, or "large perturbations" in the flow. As is evident from the preceding analysis, these nonlinearities typically become important with increasing flow turning angle and with decreasing Mach number; that is, for large geometric perturbations (through surface slope or motion) and at low supersonic speeds (due to the sensitivity of the flow to derivates in the x-direction in the transonic limit). The nonlinearity in the z-direction derivatives observed at high Mach numbers for even slender bodies is characteristic of hypersonic flows; the large perturbation results due to the high Mach number of the flow, rather than the sensitivity of the solution to high-order terms in θ. The nonlinearity through higher-order terms may affect the validity of the law of plane sections for bodies which are no longer slender – in this regard, the reader is referred to the work of Voevodenko and Panteleev [20]; within the validity of the law of plane sections, the validity of a frequency-domain solution of linearized flow is breached when higher-order terms are important.

4.3 Local piston theory

The assumption of small perturbations is central to the notion of local piston theory (LPT); linear piston theory is used to model the perturbations about a mean steady flow. In applying LPT, the reference Mach number M becomes the local Mach number of the unperturbed mean steady flow; the flow turning angle measures the perturbation about the mean steady geometry. As may be deduced from the preceding analysis, the strength of a perturbation is a function of the Mach number and the flow turning angle, and for small perturbations may be taken as $K = M\theta$, where M and θ are defined as above for LPT. The dynamic linearization of the flow and independence of the solution on streamwise influence also implies that the reduced frequency k of the flow is small. Under these conditions, the analysis developed in the present work may be used as a guideline for estimating the validity range of local piston theory. The use of LPT with viscous corrections or with CFD is outside the scope of the present analysis; however, it is expected that the validity of viscous LPT will require weak viscous interaction and the preclusion of flow separation.

5 CONCLUSIONS

The theoretical basis of piston theory and of related aerodynamic methods has been reviewed and used to provide a quantitative estimate for the validity range of linear piston theory. The work stands as a complement to similar categorizations of method validity ranges in literature, with extension to potential flow for $M > 1$ and small reduced frequency k. The use of linearized potential flow in the development of the analysis provides a consistent analytical validity range for recent extensions to piston theory. Nonlinearity is shown to be important for moderate deflections ($\theta > 5^{\circ}$) at all Mach numbers. Nonlinearity in the x-direction dominates in the transonic limit, whilst the hypersonic limit produces nonlinearities in both directions. The validity range of a transonic extension to the linearized equation is provided; challenges in extension to hypersonic flows are highlighted. The basis of local piston theory as a perturbation flow is used in guiding the estimation of the validity limits for the theory.

6 ACKNOWLEDGEMENTS

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