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A VARIATIONAL APPROACH FOR THE DYNAMICS OF UNSTEADY POINT VORTICES

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Abstract: A Lagrangian formulation for the dynamics of unsteady point vortices is proposed. This Lagrangian is shown to be equivalent to the previously constructed Lagrangian in terms of yielding exact same dynamics for vortices of constant strength. However, different dynamics is obtained in the case of unsteady point vortices. The resulting Euler-Lagrange equation derived from the principle of least action based on the proposed Lagrangian exactly matches the Brown-Michael evolution equation for unsteady point vortices, which was derived from a completely different point of view that was based on conservation of linear momentum. The resulting dynamic model of time-varying vortices is applied to two cases of unsteady point vortices, namely the starting vortex and the vortex generated by a pitching flat plate. Validation of the results of the proposed Lagrangian are determined by comparing resulting aerodynamic coefficients with those of other models and experiments.

1 INTRODUCTION

Reduced-order modeling of unsteady aerodynamics has been a topic of research interest resulting in the consequent formulations of Prandtl [1] and Birnbaum [2], Wagner [3], Theodorsen [4], Leishman [5, 6] and Peters [7, 8]; and the more recent models of Ansari et al. [9, 10], Taha et al. [11] and Yan et al. [12] among others. Another significant approach that has been taken for unsteady aerodynamic modeling is the unsteady vortex lattice method (UVLM) [13–19] or the discrete vortex method (DVM) [20]. Although this method is used to develop efficient numerical algorithms to solve for aerodynamic quantities associated with unsteady maneuvers, it requires shedding point vortices at each time step, which increases the number of degrees of freedom considerably as the simulation time increases. As a remedy, it has been suggested to replace the continuous shedding of constant-strength point vortices with discontinuous/intermittent shedding of varying-strength point vortices.

One issue that is associated with varying-strength (unsteady) point vortices is the non-uniqueness of their dynamics. In particular, they cannot convect with the Kirchhoff velocity because this will lead to spurious forces on the branch cut between the point vortex and the shedding edge. In other words, the linear and angular momenta may not be conserved, as pointed in Prandtl's lectures [25] and analyzed by Brown and Michael [26], and independently noted by Edwards [27]. Brown and Michael [26] proposed a general model for the dynamics of unsteady point vortices shed from sharp edges that removes the spurious force resulting from the time-derivative of the vortex strength. Later on, Cheng [28] and Rott [29] introduced the same concept for two dimensional flows with vortices of variable strengths. This model in conjunction with the above intermittent shedding criterion constituted the basis for the more recent efforts on reduced-order modeling of unsteady aerodynamics of maneuvering airfoils by Cortelezzi and Leonard [24] and Michelin and Smith [30].

Recently, Tchieu and Leonard [32] proposed an alternative model to Brown-Michael's for the dynamics of unsteady point vortices. The model sets the convection velocity of the unsteady vortex such that the resulting impulse is the same as that of a surrogate constant-strength vortex moving with the Kirchhoff velocity. They applied it to the problem of impulsively started flat plate and showed that the model results in a lift behavior that is closer to Wagner's lift [3] than that of Brown-Michael's [26]. Wang and Eldredge [33] generalized the model proposed by Tchieu and Leonard [32] and named it the impulse matching model. They applied this model to the cases of pitching and perching of flat plate. Both the Brown-Michael model and the impulse matching model are intrinsically concerned only with conservation of the linear momentum, i.e. they permit unbalance of the angular momentum [32, 33].

Variational principles have been shown to be useful physical-based approaches for deriving governing equations of both solids and fluids [36, 37]. These equations are obtained by setting the first variation of the action, which is the time integral of a candidate Lagrangian function, to zero. For the vortex motion, Bateman [39], followed by Serrin [41], showed that the equations of motion of vortex lines could be obtained from a variational approach with the ability to regularize the infinite velocity at the vortex center (Sec. 4 in Ref. [39]). These variational principles were also used to derive governing equations for the cases of fluid motion with distributed vorticity [42] or point vortices [43] with no boundaries, and for the case of a fluid-body interaction [44] that considered constant strength vortices only. These advances point to the possibility of developing a variational principle governing the dynamics of unsteady point vortices interacting with a circular cylinder or a body conformal to it (e.g., airfoil), which is the objective of this work. Such a formulation will allow satisfaction of conservation laws by adding constraints to the variational problem. In addition, it will enable compact and efficient coupling with other variational principles governing rigid body and structural dynamics for coupled unsteady flight dynamics and/or aeroelastic analysis. To date, there have been no developments for variational principles governing the dynamics of unsteady point vortices interacting with solid bodies enclosed by a non-zero total circulation.

In the present work, we present a new Lagrangian function for the dynamics of point vortices that is more general than Chapman's [45]. We examine the relation between the proposed Lagrangian and Chapman's Lagrangian for the cases of constant strength and time-varying point vortices. We compare the derived equations of motion to the governing equations derived by other approaches such as the Biot-Savart law for the case of constant strength vortices and the Brown-Michael model [26, 50], and the impulse matching model [32, 33] for the time-varying vortices. We apply the resulting dynamic model of time-varying vortices to the problem of an impulsively started flat plate and compare it to the numerical results obtained using UVLM. Also, we discuss coupling the variational principle for unsteady point vortices with other variational principles governing body and/or structure dynamics for aeroelastic and/or flight dynamics applications.

2 LAGRANGIAN DYNAMICS OF POINT VORTICES

2.1 Proposed Lagrangian of Point Vortices

We postulate a new Lagrangian function for the motion of point vortices in an infinite fluid in the z-plane in the most basic form as

$$L(z_k, z_k^*, \dot{z}_k, \dot{z}_k^*) = \frac{1}{i}\sum_{k=1}^{n}\Gamma_k z_k^* \dot{z}_k + W \tag{1}$$

where the first term is the bilinear function in variables z_k and $\dot{z}_k$, and the second term is the Routh stream function $W = -\frac{1}{2\pi}\sum_{k,l,k\neq l}\Gamma_k\Gamma_l \ln(z_k - z_l)(z_k - z_l)^*$. It has to be pointed out that the variable z_k and its conjugate z_k^* are treated as an independent variables. The bilinear nature of the first term ensures that the resulting equations of motion will involve only time derivatives of the first order. The same concept was introduced by Chapman whose Lagrangian is written as

$$\begin{aligned} L'(z_k, z_k^*, \dot{z}_k, \dot{\bar{z}}_k) &= \frac{1}{2i}\sum_{k=1}^{n}\Gamma_k(z_k^*\dot{z}_k - z_k\dot{\bar{z}}_k) - \frac{1}{2\pi}\sum_{k,l,k\neq l}\Gamma_k\Gamma_l ln(z_k - z_l)(z_k - z_l)^* \\ &= I_o + W \end{aligned} \tag{2}$$

where I_0 is one of the constants of motion associated with the motion of vortices of constant strengths in an infinite fluid. Chapman's Lagragian has been used in different contexts [51, 52].

It is interesting to note that the proposed Lagrangian L and Chapman's Lagrangian are related via a gauge symmetry for the case of constant-strength vortices. That is, we have

$$L' = L - \frac{1}{2i}\frac{d}{dt}\sum_{k=1}^{n}\Theta_k \tag{3}$$

where $\Theta_k = \Gamma_k z_k^* z_k$ is the angular momentum of the k^{th} vortex about the origin. Note that the gauge symmetry between any two Lagrangian functions such as L and L' implies that they add up to a total time derivative of some function, i.e., we have

$$L' = L + \frac{d}{dt}[F(q, t)]$$

where q are the generalized coordinates. As such, it is said that L and L' are related by a gauge symmetry or a gauge transformation and that both are gauge invariant [53, 54].

On the other hand, using Eq. (3), one may explain Chapman's Lagrangian L' as a constrained version of our proposed Lagrangian L to satisfy the constraint that the total angular momentum of the vortices about origin is conserved; i.e., $\frac{d}{dt}\sum_{k=1}^{n}\Theta_k = 0$.

2.2 Dynamics of a Constant Strength Point Vortices

To obtain the equations of motion for the case of vortices of constant strength, we define the action to be the integral of the Lagrangian

$$S = \int_{t_1}^{t_2} L(z_k, z_k^*, \dot{z}_k, \dot{z}_k^*) dt \tag{4}$$

Applying the principle of least action, i.e., setting the first variation of the action integral S to zero, the corresponding Euler-Lagrange equation, which is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_k}\right) - \frac{\partial L}{\partial z_k} = 0 \ , \tag{5}$$

yields the Biot-Savart law [19, 55, 56] that governs the motion of point vortices and is given by

$$\dot{z}_k^* = \frac{1}{2\pi i} \sum_{k,l,k\neq l} \frac{\Gamma_j}{z_k - z_l} \tag{6}$$

It should be noted that the same result can be obtained using Chapman's Lagrangian L' [45].

2.3 Dynamics of Unsteady Point Vortices Interacting with a Conformal Body

For a single point vortex of constant strength Γ, the Lagrangian proposed in Eq. (1) is written as

$$L(z, z^*, \dot{z}, \dot{z}^*) = \frac{1}{i}\Gamma z^* \dot{z} + W(z, z^*) \tag{7}$$

where $W(z, z^*)$ is the Kirchhoff-Routh function, which is a measure of the instantaneous energy in the flow [57] while accounting for the presence of the body. Allowing for a time-varying vortex strength (i.e. $\Gamma = \Gamma(t)$), a term that depends on the time rate of change of circulation (i.e. $\dot{\Gamma}$)) is added to ensure that the derivatives resulting from the bilinear function are coordinate-independent. As such, the Lagrangian is written as

$$L(z, z^*, \dot{z}, \dot{z}^*) = \frac{1}{i}\left(\Gamma z^* \dot{z} + \dot{\Gamma} z_0^* z\right) + W(z, z^*) \tag{8}$$

where z_0 is the coordinate of an arbitrary point on the body as shown in Figure 1. The Lagrangian of n point vortices of time-varying strengths is then written as

$$L(z_k, z_k^*, \dot{z_k}, \dot{z_k}^*) = \frac{1}{i}\sum_{k=1}^{n}\left(\Gamma_k z_k^* \dot{z_k} + \dot{\Gamma} z_{0k}^* z_k\right) + W(z_k, z_k^*) \tag{9}$$

where z_{0k} is the coordinate of a reference point on the body, which is usually the coordinate of the edge from which the vortex is shed [26, 30, 32, 33].

Applying Euler-Lagrange equations (5) associated with minimizing the action integral based on this transformed Lagrangian (9), we obtain the dynamics of an unsteady point vortex as

$$\dot{z_k} + \frac{\dot{\Gamma}_k}{\Gamma_k}(z_k - z_{0k}) = (\frac{i}{\Gamma_k}\frac{\partial W}{\partial z_k})^* \tag{10}$$

which reduces to the Biot-Savart law given by Eq. (6) if $\dot{\Gamma}$ is set to zero.

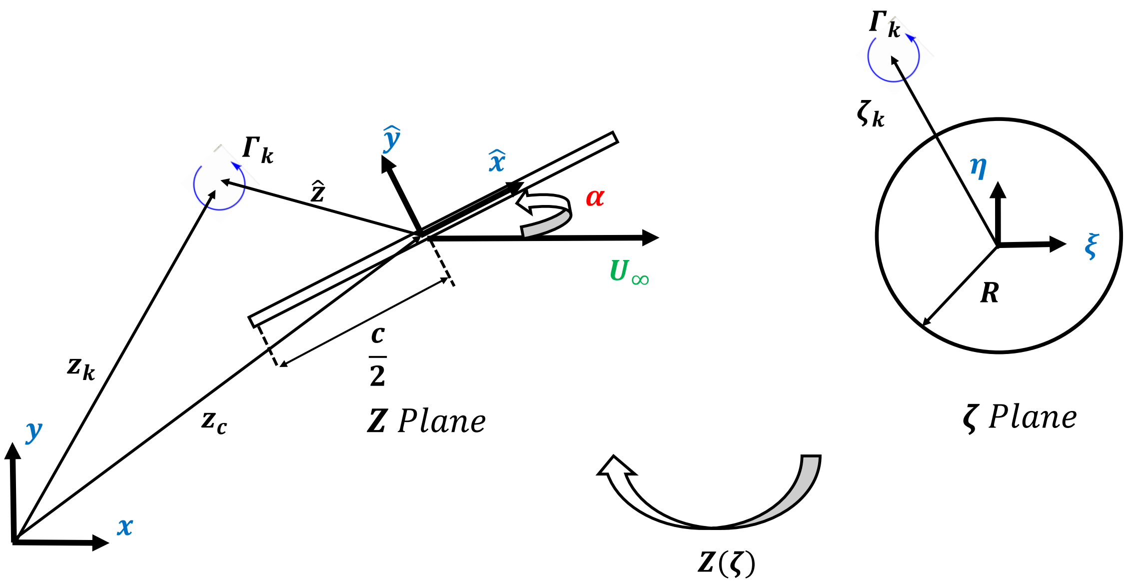


Figure 1: Conformal mapping between a sharp-edged body and a circular cylinder.

The right hand side of Eq. (10) can be represented in terms of the regularized local fluid velocity (Kirchoff velocity) $w^*(z_k)$, as shown by [56], which is expressed as

$$(\frac{i}{\Gamma_k}\frac{\partial W}{\partial z_k})^* = w^*(z_k) \tag{11}$$

Combining Eq. (10) and Eq. (11), we write

$$\dot{z}_k + \frac{\dot{\Gamma}_k}{\Gamma_k}(z_k - z_{0k}) = w^*(z_k) \tag{12}$$

which is exactly the same equation obtained by Brown and Michael [26] who used a completely different approach that was based on the conservation of linear momentum.

It is interesting to note that while both the proposed Lagrangian L and Chapman's L' [45] yield the exact same dynamics for constant-strength vortices, (the Biot-Savart law) they yield different dynamics for unsteady point vortices. Adding a similar term to Chapman's Lagrangian L' to obtain a coordinate-independent expression for the vortex absolute velocity and minimizing the action integral based on this transformed Lagrangian, the resulting equation of motion is

$$\dot{z}_k + \frac{\dot{\Gamma}_k}{2\Gamma_k}(z_k - z_{0k}) = w^*(z_k) \tag{13}$$

which differs from that of Brown-Michael by the factor of one half that multiplies the $\dot{\Gamma}$-term.

Next, we apply the variational principle approach as defined above and evaluate the performance of both postulated and Chapman's [45] Lagrangians in predicting flow quantities. For validation purposes, we compare time histories of the circulation and lift coefficient to those obtained using the impulse matching model by Wang and Eldredge [33] and Wagner's function [3].

3 IMPULSIVELY STARED FLAT PLATE (THE STARTING VORTEX PROBLEM)

We consider a flat plate of semi-chord $c/2$ mapped from a circle of radius R, as shown in Fig. 1, according to the conformal mapping

$$z(\zeta) = z_c + g(\zeta)e^{i\alpha} \tag{14}$$

where the mapping function, g, is defined as

$$g(\zeta) = \zeta + \frac{R^2}{\zeta} \tag{15}$$

The derivative of z with respect to ζ is

$$\frac{dz}{d\zeta} = g'(\zeta)e^{i\alpha} \tag{16}$$

We also consider the case where the flat plate is moving with a constant speed U_∞ , inclined to the x-axis by an angle α. A vortex of strength Γ_v is shed from the trailing edge as shown in Fig. 1. For this flow, the complex potential in the circle plane is written as [30, 56, 58]

$$F(\zeta) = \phi(\zeta) + i\psi(\zeta) = V(\zeta - g(\zeta)) + \frac{R^2\bar{V}}{\zeta} + \frac{\Gamma_v}{2\pi i}\left[ln(\zeta - \zeta_v) - ln(\zeta - \zeta_v^{(I)})\right] \tag{17}$$

where ϕ is the velocity potential, ψ is the stream function, $V = -U_\infty e^{i\alpha}$ is the velocity of the flat plate in the plate-fixed frame, and $\zeta_v^I = R^2/\zeta_v^*$ denotes the position of the image vortex within the circle. The first term inside the brackets $(\zeta - g(\zeta))$ ensures that the complex potential will contain only ζ with negative power (see Sec. 9.63 [56] , Sec. 4.71 [55], Sec. 4 [58], Sec. 3.2 [30]).

3.1 Dynamics of the Starting Vortex

Taking the origin at the mid-chord point and assuming that the starting vortex shed from the trailing edge ($\hat{z}_{v0} = -c/2$), we write the evolution equation of the starting vortex according to the Lagrangian dynamics as

$$\begin{aligned} \dot{z}_v + \frac{\dot{\Gamma}_v}{\beta\Gamma}(z_v - z_{v0}) &= \left(\frac{i}{\Gamma}\frac{\partial W}{\partial z_v}\right)^* \\ &= \left(\frac{i}{\Gamma}\frac{\partial W}{\partial \zeta_v}\left(\frac{dz}{d\zeta}\right)^{-1}_{z_v}\right)^* \\ &= w^*(z_v) \end{aligned} \tag{18}$$

where β is a factor used to differentiate between the equation obtained from the proposed Lagrangian L ($\beta = 1$) or Chapman's Lagrangian L' ($\beta = 2$). Also, we have

$$W(z_v) = \Gamma_v\psi_o + \frac{\Gamma_v^2}{4\pi}|\ln(\zeta_v - \zeta_v^{(I)})| + \frac{\Gamma_v^2}{4\pi}\ln|\frac{dz}{d\zeta}|_{z_v} \tag{19}$$

and

$$\psi_0 = Im\left(V(\zeta - g(\zeta)) + \frac{R^2V^*}{\zeta}\right) \tag{20}$$

Transforming Eq. (18) to the circle plane, the first term in the left hand side is written as

$$\dot{z}_v = U_\infty + g'(\zeta_v)e^{i\alpha}\dot{\zeta}_v \tag{21}$$

and the right hand side of Eq. (18) is re-written as

$$\begin{aligned} w^*(\zeta) =& \frac{e^{i\alpha}}{[g'(\zeta)]^*}\left[V(1-g'(\zeta)) - \frac{R^2\bar{V}}{\zeta^2} - \frac{\Gamma_v}{2\pi i}\frac{1}{\zeta-\zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta)}{g'(\zeta)}\right]^* \\ =& \frac{e^{i\alpha}}{[g'(\zeta)]^*}\left[V - \frac{R^2\bar{V}}{\zeta^2} - \frac{\Gamma_v}{2\pi i}\frac{1}{\zeta-\zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta)}{g'(\zeta)}\right]^* - Ve^{-i\alpha} \end{aligned} \tag{22}$$

Recalling that $V = -U_\infty e^{i\alpha}$, we write

$$w^*(\zeta) = \frac{e^{i\alpha}}{[g'(\zeta)]^*}\left[V - \frac{R^2\bar{V}}{\zeta^2} - \frac{\Gamma_v}{2\pi i}\frac{1}{\zeta-\zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta)}{g'(\zeta)}\right]^* + U_\infty \tag{23}$$

The evolution equation is then re-written in terms of the circle-plane variables as

$$\dot{\zeta}_v + \frac{\dot{\Gamma}_v}{\beta\Gamma_v}\frac{(g(\zeta_v)-2R)}{g'(\zeta_v)} = \frac{1}{g'(\zeta_v)[g'(\zeta_v)]^*}\left[V - \frac{R^2\bar{V}}{\zeta_v^2} - \frac{\Gamma}{2\pi i}\frac{1}{\zeta_v-\zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta_v)}{g'(\zeta_v)}\right]^* \tag{24}$$

A more general form of Eq. (24), for $\beta = 1$, for a flat plate moving and rotating in space can be found in Michelin and Smith [30].

4 NUMERICAL RESULTS

Next, we implement the proposed Lagarngian to two problems, namely the starting vortex and the vortex generated by a pitching plate. In the integration of the equations of motion, we used the Matlab solver **ode23s** with a fixed time step of $\Delta t = 10^{-5}c/U_\infty$. This solver showed a better performance than others because of the stiff nature of the evolution equation. For the first time step, instead of integrating the equations of motion analytically along with the Kutta condition as in Refs [30, 59], we used an appropriate initial condition for the position of the vortex , i.e. $x(0) = c/2 + \epsilon$, where $\epsilon \approx 10^{-4}c$.

4.1 Impulsively Started Flat Plate

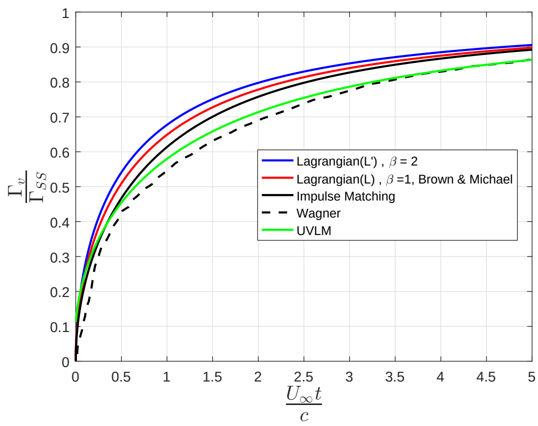
First, similarly to the classical unsteady thin airfoil theory (e.g., Wagner [3], Theodorsen [4], and Von Karman and Sears [60]), we assume that the starting vortex moves along the x-axis and the local fluid velocity is U_∞ (i.e., $w(z_v) = U_\infty$). As such, the evolution equation (18) in the z planes is given by

$$\dot{x}_v + \frac{\dot{\Gamma}_v}{\beta\Gamma_v}(x_v - x_{v0}) = U_\infty \tag{25}$$

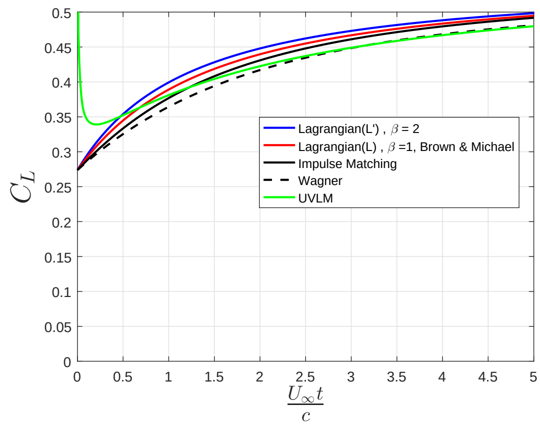
The evolution equation of the impulse matching model [32, 33] can also be simplified to

$$\dot{x}_v + \frac{\dot{\Gamma}_v}{\Gamma_v}\frac{(x_v^2 - x_{v0}^2)}{x_v} = U_\infty \tag{26}$$

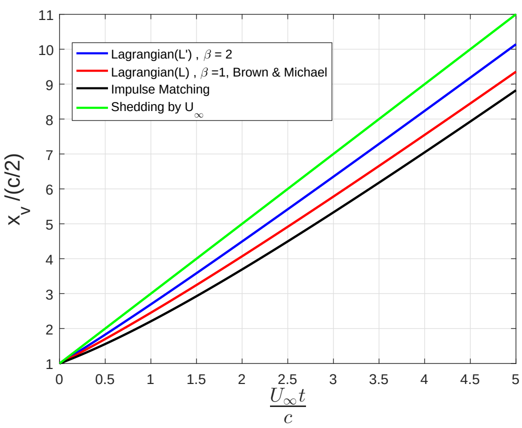
Figure [2] shows the time variations of the normalized vortex strength Γ, the lift coefficient C_L, and the time-variation of the normalized vortex location x for the case of $\alpha = 5^o$. Plots from simulations based on (i) the proposed Lagrangian dynamics ($\beta = 1$ Brown-Michael),



(a) Time variation of the normalized vortex strength Γ_v. The circulation is normalized with the steady-state value Γ_{SS}.



(b) Time variation of the lift coefficient C_L

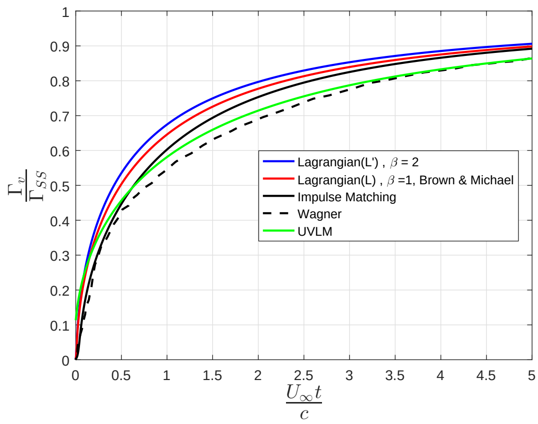


(c) Time variation of the normalized vortex position x_v. The position is normalized using the semi-chord of the airfoil.

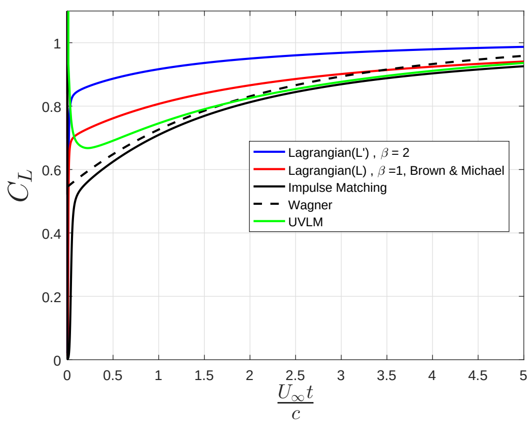
Figure 2: Time variations of (a) the normalized circulation, (b) lift coefficient and (c) normalized position of the starting vortex for $\alpha = 5^o$ and the vortex is assumed to move only in the x direction. The time is normalized using the airfoil speed U_∞ and chord c.

(ii) Chapman's Lagrangian ($\beta = 2$), (iii) the impulse matching model of Wang and Eldredge [33], (iv) Wagner's [3] step response function, (v) and the UVLM are presented for the sake of comparison. The plots show that all models agree qualitatively with Wagner's exact potential flow solution with the UVLM showing the best agreement. Note that in the three models, the infinite sheet of wake vorticity is approximated by a single vortex. On the contrary, in UVLM model, a vortex is shed at each time step and that vortex is allowed to move in the plane. As expected, the correction to the Kirchhoff velocity (taken as U_∞ here) in the case of $\beta = 2$ is half of that in the case of $\beta = 1$ yields slightly higher transient lift.

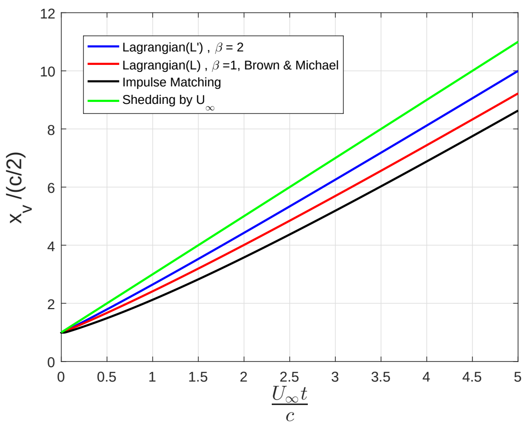
Next, we consider increasing the angle of attack to $\alpha = 10°$ to relax the flat wake assumption. Thus, allowing the vortex to move in the plane, i.e. with two degrees of freedom. Figure [3] shows the resulting time variations of the normalized circulation Γ, lift coefficient C_L, vortex position along the x-axis, and the slope of the vortex trajectory θ as a function of x. The singular value of the lift at $t = 0$, which corresponds to the added mass effect, is removed to highlight the difference between results from different models. Again, the results based on L' ($\beta = 2$) predict a larger vortex strength (airfoil circulation) and a slightly higher lift, than those predicted by the two other models. Figure [3d] shows that the slope of the starting vortex asymptotically approaches a line parallel to the incident free stream (i.e. $\theta \approx \alpha = 10^o$). As shown, the proposed Lagrangian (Brown-Michael model) yield lift and circulation values that do not match Wagner's function. In addition, the impulse matching results in a slower downstream convection. Consequently the development of circulation takes place at a slower rate with an overall effect of reduced lift coefficient that matches Wagner's function. We note, however, that the Wagner's response should not be considered as a reference for comparison in this case because of the flat-wake and shedding by U_∞ assumptions that may not be appropriate for this relatively high angle of attack.



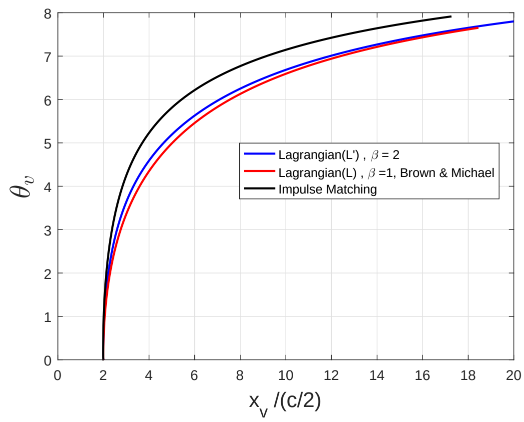
(a) Time variation of the normalized vortex strength Γ_v. The circulation is normalized with the steady-state value Γ_{SS}



(b) Time variation of the lift coefficient C_L



(c) Vortex position x_v versus non-dimensional time



(d) Slope of the vortex trajectory θ_v versus vortex position x_v

Figure 3: Time variations of (a) the normalized circulation, (b) lift coefficient, (c) normalized position of the starting vortex, and the slope of the vortex trajectory for $\alpha = 10^o$ and the vortex is allowed to move freely in the plane of the airfoil. The time is normalized using the airfoil speed U_∞ and chord c.

In Figure [4], the lift coefficient versus angle of attack is shown for an airfoil pitching at a reduced frequency $k = 0.2$, and compared to the experiment carried out by Granlund et al. [34] at Reynolds number $Re = 20,000$. In this case, two vortices are shed from the leading and trailing edges. The same trend as in the case of the starting vortex is noted. Moreover, the difference is maximum when the angle of attack reaches 45^o and approaches zero when the angle of attack reaches 90^o. We also noted that while both proposed and Chapman's Lagrangian yielded similar dynamics for the case of the starting vortex, they yielded different dynamics for the case of pitching flat plate. The proposed Lagrangian (Brown -Michael) yielded a better agreement with the experimental results than Chapman's Lagrangian.

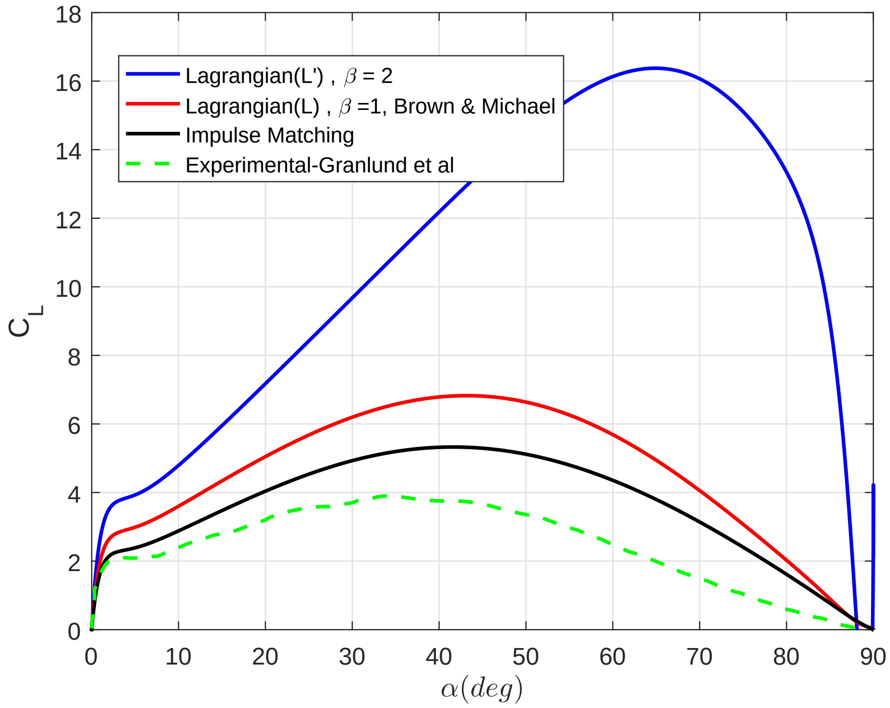


Figure 4: Lift coefficient versus angle of attack for pitching airfoil at reduced frequency $k = 0.2$ and Reynolds number $Re = 20,000$.

5 POTENTIAL AND FUTURE ADVANCEMENTS

The main contribution of this effort is providing a successful Lagrangian function that governs the dynamics of unsteady point vortices. Having this Lagrangian invokes the development of variational principles that govern flight dynamics and/or aero-elastic systems. There have been several successful variational principles governing structure dynamics (e.g., the principle of minimum potential energy). The dynamics of the aeroelastic system is typically written as

$$\frac{d}{dt}\left(\frac{L_{tot}}{\partial \dot{q_s}}\right) - \frac{\partial L_{tot}}{\partial q_s} = Q \tag{27}$$

where q_s are the structural generalized coordinates, L_s is the Lagrangian function of the structural system, and Q represents the non-conservative applied loads. In this typical formulation, the aerodynamic loads (of unknown nature) are incorporated in the right hand side as non-conservative loads; due to the lack of an *aerodynamic* Lagrangian and/or variational principle for unsteady fluids even within the framework of potential flow.

Using the proposed Lagrangian L for unsteady aerodynamics, we can, for the first time, write a *single* Lagrangian L_{tot} governing the dynamics of the whole aero-elastic system; providing a single variational principle for both the fluid flow and structure, which has been the subject of interest for decades [61]. As such, the aerodynamic loads will be naturally accounted for in a similar fashion to the structural restoring forces in the left hand side of Lagranges equations

$$\frac{d}{dt}\left(\frac{L_{tot}}{\partial \dot{q}}\right) - \frac{\partial L_{tot}}{\partial q} = 0 \tag{28}$$

where $q = [qs\ qa]$ and qa represents the generalized coordinates of the aerodynamic system (e.g., position and strength of the shed vortices). The variational equation (28) will invoke discovery of conserved quantities and more compact analysis of aeroelastic systems.

6 CONCLUSIONS

We investigated the potential of implementing variational principles to derive governing equations for the interaction of unsteady point vortices with a solid boundary. To do so, we postulated a new Lagrangian function for the dynamics of point vortices that is more general than Chapman's. We showed that this function is related to Chapman's Lagrangian via a gauge symmetry for the case of constant-strength vortices. In other words, both Lagrangian functions result in the same governing equation, i.e. the Biot-Savart law is directly recovered from the Euler-Lagrange equations corresponding to minimization of the action integral with these two Lagrangians. We also found that, unlike Chapman's Lagrangian, the principle of least action based on the proposed Lagrangian results exactly in the Brown-Michael model for the dynamics of unsteady point vortices. We implemented the resulting dynamic model of time-varying vortices to the problems of the staring vortex and pitching airfoil and compared the results to those of the Wagner solution, UVLM, and experiments from the literature. The results showed that the proposed Lagrangian yields better agreement than Chapman's with numerical and experimental results.

ACKNOWLEDGMENTS

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