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SIMULATION-FREE HYPER-REDUCED MODELS FOR GEOMETRICALLY NONLINEAR STRUCTURAL DYNAMICS

S. Jain[1] and P. Tiso[1]

[1]Institute for Mechanical Systems, ETH Zürich
Leonhardstrasse 21, 8092 Zürich, Switzerland
{ptiso}{shjain}@ethz.ch

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Abstract: We present a variant of the Energy Conserving Sampling and Weighting (ECSW) method that minimizes the offline cost associated to the construction of reduced order models of FE discretized geometrically nonlinear structural dynamics problems. The training sets required by ECSW are obtained by lifting linear modal analysis responses on a quadratic manifold generated with modal derivatives. The resorting on expensive POD of the full response is completely avoided. The method is particularly suited for the dynamic analysis of flexible airframes featuring geometric nonlinearities.

1 INTRODUCTION

Detailed Finite Elements (FE) models of ariframe structures often tend to possess large number of degrees of freedom (DOFs) in order to account for extremely detailed geometric features and material distribution. However, routine simulations to explore different load scenarios, geometric layouts and material choice for such large models carry prohibitive computational costs. In this context, reduced-order models (ROMs) offer a reprieve from such extreme computational costs and allow effective design and optimization activities.

In a broad sense, ROMs are low-dimensional counterparts of the original model, often referred as high-fidelity model (HFM). Classically, this reduction is achieved through a linear projection of the full system of equations onto a reduction basis, which has been precomputed *offline*. This basis spans a low-dimensional invariant subspace suitable for capturing the HFM solution. As a result, the ROM is characterized by only a few DOFs. The construction cost of the reduced nonlinear operators (i.e. internal elastic forces and tangent stiffness matrix) in the projected equations, however, scales with the size of the HFM ans not with that of the ROM. This constitutes a serious bottle-neck for achieving significant speed-ups, as the assembly of these terms dominates the cost of the time integration. Hence, such ROMs are effective only if these reduced operators can be precomputed offline.

More often than not, nonlinear modeling is essential for design and analysis of realistic structures, even in the preliminary stages. Thin-walled structural components, for instance, are typically employed in the aerospace industry when high stiffness-to-weight and strength-to-weight ratios must be achieved. Among other nonlinear effects, the geometric nonlinearities are particularly important in modeling their behavior, including peculiar effects like bending-stretching coupling; buckling; snap-through; mode jumping etc., due to finite rotations [8]. This has given rise to a pressing need for effective reduction of large nonlinear dynamical systems. To this effect, various techniques have been developed over the recent years, which have made the *online* computation of the reduced nonlinear operators in ROMs, much cheaper [1, 3–6]. These are commonly referred to as *hyper-reduction* techniques.

Generally, hyper-reduction techniques alleviate the computational costs of the nonlinear terms by optimally selecting a small set of nodes (or elements) in the mesh over which the nonlinearity is evaluated. The nonlinearity is then cheaply interpolated over the rest of the mesh. This selection process is performed with the help of *training* vectors. These training vectors are usually obtained from the solution of the HFM. Such *full*-solution vectors are often also used for the construction of the reduction basis used in projection-based ROMs, for example, using the proper orthogonal decomposition (POD) [12–14]. The use of these full-solution *snapshots* for training and reduction purposes is a computationally expensive affair, which can be unaffordable, especially at the preliminary design stage of structures.

Many techniques enable the construction of a ROM or reduction basis for nonlinear problems without the need of a full solution (cf. [15, 16] more refs.). Furthermore, for a certain class of problems–characterized by mild geometric nonlinearities–the use of vibration modes (VMs) and modal derivatives (MDs) has been shown to be effective for construction of a modal-based reduction basis [9–11]. Indeed, the robustness and effectiveness of full-simulation vectors in general nonlinear reduction scenarios (e.g. using POD) cannot be discounted. Nonetheless, such modal-based reduction techniques, albeit being sub-optimal, find use when the available time and computational resources do not allow for creating a database of full simulation runs. By avoiding full solutions, these techniques go a long way in reducing offline costs incurred during construction of an effective reduction basis.

The generation of training vectors for hyper-reduction of nonlinear terms in the ROM, however, still leads to tremendous offline costs if they are still based on full-simulations. To this effect we propose a modal-based training-set generation technique, which completely avoids the HFM simulations, thereby, making the reduction procedure truly *simulation-free.*

The modal derivatives have been classically used to form a reduction basis along with VMs to capture geometrically nonlinear behavior. Recently, they were also used for reduction via a quadratic manifold [9], where a linear subspace, formed by a truncated set of VMs, captures the linearized dynamics near the equilibrium and the corresponding MDs provide the necessary nonlinear (quadratic) extension to this subspace. In this work, we use this notion of a quadratic manifold to generate meaningful training vectors from a linear modal superposition of the underlying linearized system. This unified approach builds on to the linear modal signature, which is rather cheaply available and essential for the analysis of any structural system. We have tested this approach on a realistic structural model of a wing. Furthermore, we have compared, for the first time, the offline costs and *effective* speedups involved in reduction and hyper-reduction with that of other established techniques, by taking into account althe effort needed to construct such a simulation-free hyper-reduced ROM.

This paper is organized as follows. In the next section, we start by reviewing the concepts of projection-based model reduction, which results in the reduction of the dimensionality of the HFM. The concept of hyper-reduction is reviewed in Section 3, where we propose the use of a stability-preserving, finite-element-based hyper-reduction technique, known as energy conserving sampling and weighing (ECSW) [1]. The modal-based training-set generation using the quadratic manifold is presented in Section 4. The numerical results for the tested example, along with comparisons with other techniques, are presented in Section 5. Finally, the conclusion are given in Section 6.

2 DIMENSIONALITY REDUCTION

The partial differential equations (PDEs) for momentum balance in a structural continuum are first FE-discretized along the spatial dimensions to obtain a system of second order ordinary differential equations (ODEs). Along with the initial conditions for generalized displacements and velocities, these ODEs govern the response of the underlying structure. More specifically, this response can be described by the solution to an initial value problem (IVP) of the following form:

$$\begin{gathered}\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t)) = \mathbf{g}(t)\,, \\ \mathbf{u}(t_0) = \mathbf{u}_0,\, \dot{\mathbf{u}}(t_0) = \mathbf{v}_0,\end{gathered} \tag{1}$$

where the solution $\mathbf{u}(t) \in \mathbb{R}^n$ is a high-dimensional generalized displacement vector, $\mathbf{M} \in \mathbb{R}^{n\times n}$ is the mass matrix, $\mathbf{C} \in \mathbb{R}^{n\times n}$ is the damping matrix, $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ gives the nonlinear elastic internal force as a function of of the displacement $\mathbf{u}$ of the structure, and $\mathbf{g}(t) \in \mathbb{R}^n$ is the time dependent external load vector. The nonlinear term $\mathbf{f}(\mathbf{u})$ models the effect of geometric nonlinearities, arising in the case of large deflections and rotations. In this work, von Karman kinematics has been used to model geometric nonlinearties for shell-based structures. As discussed in Section 1, the system (1) is referred to as the HFM. The response of the HFM can be extremely time consuming to compute if the dimension n of the system is large. The classical notion of model reduction aims to reduce this dimensionality by introducing a linear mapping on to a suitable low-dimensional invariant subspace $\mathcal{V}$ as

$$\mathbf{u}(t) \approx \mathbf{V}\mathbf{q}(t)\,, \qquad \mathbf{V} \in \mathbb{R}^{n\times m},$$

where $\mathbf{q}(t) \in \mathbb{R}^m$ ($m \ll n$) is the low-dimensional vector of reduced variables, and $\mathbf{V}$ is known as the reduction basis since its columns form a basis for $\mathcal{V}$. The reduced-order model is then obtained using Galerkin projection as

$$\underbrace{\mathbf{V}^T\mathbf{M}\mathbf{V}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}}(t) + \underbrace{\mathbf{V}^T\mathbf{C}\mathbf{V}}_{\tilde{\mathbf{C}}}\dot{\mathbf{q}}(t) + \mathbf{V}^T\mathbf{f}(\mathbf{V}\mathbf{q}(t)) = \mathbf{V}^T\mathbf{g}(t),$$

where $\tilde{\mathbf{M}}$, $\tilde{\mathbf{C}} \in \mathbb{R}^{m\times m}$ are the reduced mass and damping matrices, respectively. Often, the internal force can be split in to its linear and nonlinear contributions as $\mathbf{f}(\mathbf{u}) = \mathbf{K}\mathbf{u} + \mathbf{f}^{nl}(\mathbf{u})$, to obtain a reduced stiffness matrix as well:

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{q}}(t) + \underbrace{\mathbf{V}^T\mathbf{K}\mathbf{V}}_{\tilde{\mathbf{K}}}\mathbf{q}(t) + \underbrace{\mathbf{V}^T\mathbf{f}^{nl}(\mathbf{V}\mathbf{q}(t))}_{\tilde{\mathbf{f}}(\mathbf{q}(t))} = \mathbf{V}^T\mathbf{g}(t). \tag{2}$$

It is easy to see that each of the reduced matrices $\tilde{\mathbf{M}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ can be precomputed in the offline stage prior to time integration. Hence, during time integration (online phase), the computational cost associated to the evaluation of the linear terms in (2) scales only with the number of reduced variables m. This is, however, not the case for the computation of the nonlinear term $\tilde{\mathbf{f}}(\mathbf{q}(t))$. For FE-based applications, this evaluation is usually carried out online in the following manner:

$$\tilde{\mathbf{f}}(\mathbf{q}) = \mathbf{V}^T\mathbf{f}^{nl}(\mathbf{V}\mathbf{q}) = \sum_{e=1}^{n_e}\mathbf{V}_e^T\mathbf{f}_e(\mathbf{V}_e\mathbf{q}), \tag{3}$$

where $\mathbf{f}_e(\mathbf{u_e}) \in \mathbb{R}^{N_e}$ is the contribution of the element e towards the vector $\mathbf{f}^{nl}(\mathbf{u})$ (N_e being the number of DOFs for the element e), $\mathbf{V}_e$ is the restriction of $\mathbf{V}$ to the rows indexed by the DOFs corresponding to e, and n_e is the total number of elements in the structure. Since the reduced nonlinear term $\tilde{\mathbf{f}}(\mathbf{q})$ is evaluated in the space of full variables, the computational cost associated to its evaluation does not scale with m only. Indeed, (3) shows that this cost scales linearly with the number of elements in the structure, and can hence be high for large systems. Thus, despite the reduction in dimensionality achieved in (2), the evaluation of the reduced nonlinear term $\tilde{\mathbf{f}}(\mathbf{q})$ emerges as a new bottleneck for the fast prediction of system response using the ROM. Hyper-reduction techniques help mitigate these high computational costs by approximation of the reduced nonlinear term in a computationally affordable manner.

3 HYPER-REDUCTION

In order to reduce the computational burden associated to the nonlinear force vector, we employ the recently proposed energy-conserving sampling and weighting (ECSW) hyper-reduction method [1], which directly approximates the reduced nonlinear term $\tilde{\mathbf{f}}(\mathbf{q})$ while preserving the symmetry of the tangential operations and thus numerical stability [2]. Essentially, ECSW aims to identify a small set of elements E of the structure ($|E| \ll n_e$) to cheaply approximate $\tilde{\mathbf{f}}(\mathbf{q})$ as (cf. (3))

$$\tilde{\mathbf{f}}(\mathbf{q}) = \sum_{e=1}^{n_e} \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e\mathbf{q}) \approx \sum_{e\in E} \xi_e \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e\mathbf{q}), \tag{4}$$

where $\xi_e \in \mathbb{R}^+$ is a positive weight attached to each element $e \in E$, which is empirically chosen to ensure a good approximation of the summation in (3). Clearly, if $|E| \ll n_e$, then the evaluation of the approximation in (4) would come at a fraction of the original computational cost associated to (3). In doing so, ECSW approximates the virtual work done by the internal force on the set of vectors in the basis $\mathbf{V}$. As a consequence, the ECSW preserves the structure and the stability properties of the underlying full model (cf. [2]).

The elements and weights are determined to approximate virtual work over chosen training sets which generally come from full solution run(s). If there are n_t training vectors in the set with $\mathbf{u}^{(i)}$ representing the i^{th} vector, then corresponding reduced unknowns $\mathbf{q}^{(i)}$ can be easily calculated using least squares as

$$\mathbf{q}^{(i)} = (\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{u}^{(i)},$$

and element level contribution of projected internal force for each of the training vectors can be assembled in a matrix $\mathbf{G}$ as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_{11} & \cdots & \mathbf{g}_{1n_e} \\ \vdots & \ddots & \vdots \\ \mathbf{g}_{n_t1} & \cdots & \mathbf{g}_{n_tn_e} \end{bmatrix} \in \mathbb{R}^{mn_t\times n_e}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n_t} \end{bmatrix} \in \mathbb{R}^{mn_t},$$

$$\mathbf{g}_{ie} = \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e\mathbf{q}^{(i)}), \qquad \mathbf{b}_i = \tilde{\mathbf{f}}\left(\mathbf{q}^{(i)}\right) = \sum_{e=1}^{n_e} \mathbf{g}_{ie}\,, \quad \forall i \in \{1,\dots,n_t\}, \quad e \in \{1,\dots,n_e\}\,. \tag{5}$$

The set of elements and weights is then obtained by a sparse solution to the following non-negative least-squares (NNLS) problem

$$(P1) : \boldsymbol{\xi} = \arg \min_{\tilde{\boldsymbol{\xi}}\in\mathbb{R}^{n_e}, \tilde{\boldsymbol{\xi}}\geq \mathbf{0}} \|\mathbf{G}\tilde{\boldsymbol{\xi}} - \mathbf{b}\|_2, \tag{6}$$

A sparse solution to $(P1)$ returns a sparse vector $\boldsymbol{\xi}$, the non-zero entries of which form the reduced mesh E used in (4) as

$$E = \{e : \xi_e > 0\}.$$

An optimally sparse solution to $(P1)$ is NP-hard to obtain. However, a greedy-approach-based algorithm [7], which finds a sub-optimal solution, has has been found to deliver an effective reduced mesh E [1].

4 SIMULATION-FREE REDUCTION

Effective reduction of nonlinear dynamical systems, involves both dimensionality reduction (usually using projection-based methods) as well as hyper-reduction to speed up the (reduced) nonlinearity evaluation. Conventionally, both these steps require HFM simulations which lead to tremendous offline costs. In this case, both the basis vectors and the training forces are obtained by a POD analysis of the simulation of the HFM. The POD basis is constructed as follows. Let $\mathbf{u}_i \in \mathbb{R}^n$, $i \in \{1, \ldots, n_t\}$ be training vectors obtained from the full solution of system (1). Let $\mathbf{U} := [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n_t}] \in \mathbb{R}^{n \times n_t}$ be the ensemble of snapshots obtained from the full solution. A lower dimensional POD basis $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_m] \in \mathbb{R}^{n \times m}$ containing $m \ll n_t$ orthogonal vectors which *best* spans the vectors in this ensemble can be obtained by the Single Value Decomposition (SVD) of the matrix $\mathbf{U}$, as

$$\mathbf{U} = \mathbf{LSR}^T, \tag{7}$$

where is $\mathbf{U}$ is factorized into unitary matrices $\mathbf{L} = [\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_n] \in \mathbb{R}^{n \times n}$ (containing the left singular vectors) and $\mathbf{R} \in \mathbb{R}^{n_t \times n_t}$ (containing the right singular vectors); and the diagonal (rectangular) matrix $\mathbf{S} \in \mathbb{R}^{n \times n_t}$ (containing corresponding singular values on the diagonal). The first vectors in $\mathbf{L}$ are the most evergetically significant modes shape that represent he corresponding snapshots, and are used to form the reduction basis. The POD technique is known to yield an optimal reduction basis for the underlying HFM simulation, but little can be said on the validity of such basis for another case of interest (as, for instance, a different load). As pointed out on the introduction, the cost of even one single HFM instance could be prohibitive or not justifiable in a preliminary design phase, where many different scenarios need to be explored. To reduce these offline costs for geometrically nonlinear thin-walled structures, we propose the following systematic procedure for obtaining a reduction basis, as well as for generating training vectors required for hyper-reduction, while completely avoiding full simulation runs.

4.1 Modal-based reduction basis

For geometrically nonlinear problems with separation in time scales between in-plane and out-of-plane behavior (beams, shells, thin plates, and their assemblage), it is well known that the VMs and MDs form a good basis to represent the nonlinear displacement field [9, 11, 17, 18]. We propose their use in construction of the required reduction basis $\mathbf{V}$ as

$$\boldsymbol{\Psi} = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_M, \ldots, \boldsymbol{\theta}_{ij}, \ldots], \qquad i, j \in \{1, \ldots, M\}, \tag{8}$$

$$\mathbf{V} = \texttt{orth}(\boldsymbol{\Psi}), \tag{9}$$

where $\mathbf{V}$ is obtained after an orthogonalization of the matrix $\boldsymbol{\Psi}$ in (8), `orth` represents any routine to orthogonalize any general matrix such that the result is a orthonormal matrix with a full column rank (e.g., the Gram-Schmidt orthogonalization). The matrix $\mathbf{\Psi}$ contains the relevant VMs and MDs, $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_M$ represent a truncated set of VMs obtained from the solution of the undamped eigenvalue problem

$$\left(\mathbf{K} - \omega_i^2\mathbf{M}\right)\boldsymbol{\phi}_i = \mathbf{0} \qquad \forall i \in \{1, \ldots, n\}, \tag{10}$$

and $\boldsymbol{\theta}_{ij}$ are the MDs, obtained from the solution to the following problem

$$\left(\mathbf{K} - \omega_i^2\mathbf{M}\right)\boldsymbol{\theta}_{ij} + \left(\left.\frac{\partial^2\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}\partial\mathbf{u}}\right|_{\mathbf{u}=\mathbf{0}}\boldsymbol{\phi}_j - \frac{\partial\omega_i^2}{\partial\eta_j}\mathbf{M}\right)\boldsymbol{\phi}_i = \mathbf{0}, \quad i, j \in \{1, \ldots., M\}. \tag{11}$$

This represents a derivative of the eigenvalue problem (10) with respect to the modal amplitude η_j of the VM $\boldsymbol{\phi}_j$, after $\mathbf{K}$ is replaced by the tangent stiffness matrix $\frac{\partial\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}}$. An effective static approximation of the MDs, obtained by neglecting the mass terms in (11) as

$$\boldsymbol{\theta}_{ij}^{static} = -\mathbf{K}^{-1}\left[\left.\frac{\partial^2\mathbf{f}(\mathbf{u})}{\partial\mathbf{u}\partial\mathbf{u}}\right|_{\mathbf{u}=\mathbf{0}}\boldsymbol{\phi}_j\right]\boldsymbol{\phi}_i. \tag{12}$$

The $\boldsymbol{\theta}_{ij}^{static}$ were first introduced in [11]), and termed as static MDs (SMDs) in [9].

4.2 Quadratic-manifold-based training set generation

To alleviate the prohibitive offline computational costs involved in the HFM-snapshots-based hyper-reduction, we seek to obtain relevant training vectors without the need of these full solution vectors. Starting with the idea of modal superposition, we propose a quadratic-manifold-based approach [9] to avoid these prohibitive costs in training.

The linear modal superposition using a few significant VMs is a well-established technique to obtain the reduced solution of a linear dynamical system. However, when the nonlinearities become significant, the modal basis can be augmented with MDs to effectively capture the response. If a light damping is assumed, the linearized solution $\mathbf{u}_{lin}(t)$ of the HFM in response to the applied external loading $\mathbf{g}(t)$ can be spectrally decomposed using linear modal superposition as (cf. [19])

$$\mathbf{u}_{lin}(t) = \sum_{i=1}^{n}\boldsymbol{\phi}_i\underbrace{\int_0^t\frac{\sin(\omega_i(t-\tau))}{\left(\boldsymbol{\phi}_i^T\mathbf{M}\boldsymbol{\phi}_i\right)\omega_i}\boldsymbol{\phi}_i^T\mathbf{g}(t)\,\mathrm{d}\tau}_{\eta_i(t)} = \sum_{i=1}^{n}\boldsymbol{\phi}_i\eta_i(t)\,, \tag{13}$$

where $\eta_i(t)$ are the time varying modal amplitudes corresponding to the VM $\boldsymbol{\phi}_i$. Depending on the spatial and temporal properties of the external load $\mathbf{g}(t)$, the summation in (13) can be truncated to obtain

$$\mathbf{u}_{lin}(t) \approx \sum_{i\in\mathcal{M}}\boldsymbol{\phi}_i\eta_i(t) = \mathbf{\Phi}\boldsymbol{\eta}(t), \tag{14}$$

where $\mathcal{M} \subset \{1, \ldots, n\}$ is the set of indices corresponding of the most significant modes in modal superposition (13).

Clearly, the snapshots of $\mathbf{u}_{lin}(t)$ would not be effective as training vectors needed for hyper-reduction. Indeed, consider, for example, a flat thin walled shell structure with some externally applied out-of-plane loading. The modal truncation would feature bending dominated modes which are out of plane for a linearized solution. The nonlinear response, however, is expected to feature in-plane effects due to bending/torsion-stretching coupling. These essential features would be missing in snapshots of $\mathbf{u}_{lin}(t)$, making them poor for training nonlinear hyper-reduction. The modal amplitudes, however, still contain important bending information of the structure and can be useful in generating effective training vectors by using the notion of a quadratic manifold, which was introduced for nonlinear model reduction in [9]. There, the full unknowns are mapped to a lower dimensional set of variables $\mathbf{z}$ using a quadratic mapping as

$$\mathbf{u}(t) \approx \mathbf{\Gamma}(\mathbf{z}(t)) := \mathbf{\Phi}\mathbf{z}(t) + \frac{1}{2}\mathbf{\Omega} : (\mathbf{z}(t) \otimes \mathbf{z}(t)), \tag{15}$$

where $\mathbf{\Phi} \in \mathbb{R}^{n \times M}$ containing a truncated set of $M(= |\mathcal{M}|)$ significant VMs, resulting from the linear modal superposition; and $\mathbf{\Omega} \in \mathbb{R}^{n \times M \times M}$ is a third order tensor containing the corresponding MDs. As discussed in [9], the tangent space of the quadratic manifold at equilibrium ($\mathbf{u} = \mathbf{0}$) is the modal subspace represented by $\mathbf{\Phi}$, which is corrected using the MD information to account for nonlinear behavior upon departure from the equilibrium. Furthermore, the MDs capture the inherent bending/torsion-stretching coupling arising from geometric nonlinearities. These components arise naturally as second components of the quadratic manifold. This provides a straight-forward method to generate physically-meaningful training vectors using the snapshots of the modal amplitudes $\boldsymbol{\eta}(t)$ in (15), as

$$\mathbf{u}^{(t)} = \mathbf{\Gamma}(\boldsymbol{\eta}(t_i)), \quad i \in \{1, \ldots, n_t\},$$

where $t_i \in \mathcal{T}$ (with $\mathcal{T}$ being the simulation time-span) are the time instants at which the modal snapshots are captured; n_t is the number of such snapshots chosen for training.

This simple criterion implicitly assumes that the linear behavior is correct up to a first order and the essential nonlinear bending/torsion-stretching coupling effects (captured using the quadratic manifold) are of second order. This does not necessarily imply that the nonlinear forces are small. In fact, the range of deflections we are interested in makes the linear and nonlinear forces comparable in accordance with the von Karman kinematics. The degree to which this concept can be stretched would be a problem-dependent issue. The advantage of this method lies in the fact that the linear modal solution is available for any given system at practically no costs, resulting in physically-relevant training vectors without the need of expensive high-fidelity simulation. Note that the training vectors thus obtained could, in principle, be used for training any hyper-reduction technique, and therefore this method is not restricted to ECSW hyper-reduction.

5 NUMERICAL RESULTS

5.1 Setup

We illustrate the performance of the proposed hyper-reduction technique on a model of a NACA-airfoil-based wing model, with realistically high number of DOFs, introduced in [9]. The details of the models are shown in Figure 1. The structure is modeled using flat, triangular-shell elements with 6 DOFs per node (i.e., 18 DOFs per element). For both the models, the accuracy of the results was compared to the corresponding full nonlinear solutions using a mass-normalized global relative error (GRE) measure, defined as

$$GRE_M = \frac{\sqrt{\sum_{t \in S} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))^T \mathbf{M} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))}}{\sqrt{\sum_{t \in S} \mathbf{u}(t)^T \mathbf{M} \mathbf{u}(t)}} \times 100,$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ is the vector of generalized displacements at the time t, obtained from the HFM solution, $\tilde{\mathbf{u}}(t) \in \mathbb{R}^n$ is the solution based on the (hyper) reduced model, and S is the set of time instants at which the error is recorded. The mass matrix $\mathbf{M}$ provides a relevant normalization for the generalized displacements, which could be a combination of physical displacements and rotations, as is the case in the shell models shown here.

Since the success of reduction techniques is often reported in terms of savings in simulation time, we define an online speedup $S^\star$ computed according to the following simple formula:

$$S^\star = \frac{T_{full}}{T^\star_{sim}},$$

where T_{full} and $T^\star_{sim}$ represent the CPU time taken during the time integration of *full* and (hyper)reduced solution respectively. The superscript $\star$ denotes the reduction technique being used. The speed-up defined in this manner takes only the online costs into account. For a more fair comparison, we take offline costs into account by defining an *effective* speed-up as

$$S^\star_{eff} = \frac{c_{on} T_{full}}{c_{off} T^\star_{off} + c_{on} T^\star_{sim}},$$

where $T^\star_{off}$ represents the computational time spent *offline* for setting up a (hyper-)reduced model; $c_{on}, c_{off} \in [0, 1]$ represent the relative weights given to the online and offline costs, respectively, such that $c_{on} + c_{off} = 1$. The HFM simulation is assumed to carry zero offline costs and, thus, $S_{eff} = S = 1$ for a full HFM solution. A higher value of S_{eff} for a given (hyper-) reduction method more favorable than those corresponding to a lower value, with $S_{eff} < 1$ implying that the corresponding reduction technique is effectively more expensive than a full solution run.

The results from the following reduction techniques are compared:

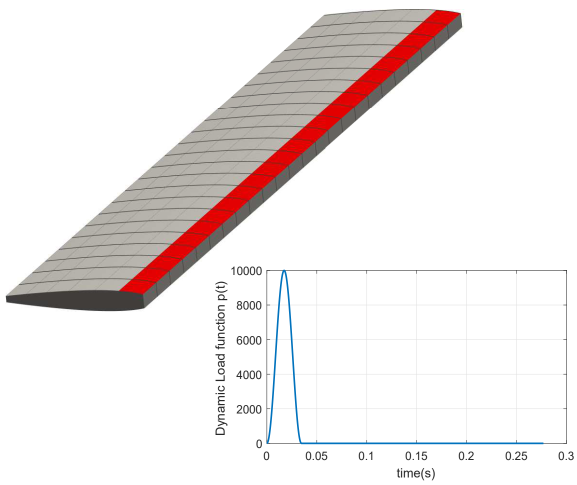
- **SIMFREE**: Simulation-free reduction (No hyper-reduction). These technique involve the simulation of the Galerkin ROM (2), where the reduction basis is obtained without the use of full simulation snapshots $\mathbf{V}$. Here, a few significant VMs (say M in number) of the structure are selected based on linear modal superposition for the given load. These VMs, along with the corresponding SMDs (which would be $\frac{M(M+1)}{2}$ in number), are used to construct $\mathbf{V}$. Thus, the size of the basis would be $m = \frac{M(M+3)}{2}$.
- **POD** (No hyper-reduction): Here, the nonlinear system using is reduced using a POD basis. The reduction basis of the same size as in Simulation-free reduction and is constructed through the SVD of the full-solution-snapshots-matrix. Thus, $m = \frac{M(M+3)}{2}$ left singular vectors with the highest singular values are included in $\mathbf{V}$.
- **POD-ECSW**: The ECSW method is used with the same basis in POD, to hyper-reduce the nonlinearity in the ROM. The HFM simulation snapshots are used in the offline stage for training to obtain the reduced mesh.
- **SIMFREE-ECSW**: Simulation-free hyper-reduction. The proposed quadratic-manifold-based training set generation is used to reduce offline costs during hyper-reduction.

5.2 Wing structure

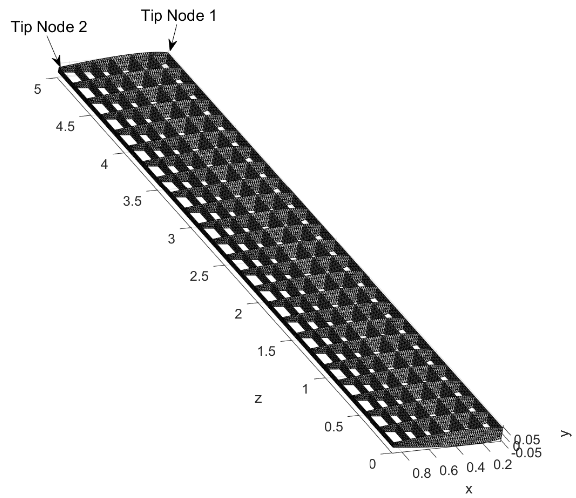
For the application of the proposed simulation-free hyper-reduction methods to more realistic models, we consider the model of a NACA-airfoil-based wing structure, introduced in [9]. This model (referred to as Model-II hereafter) contains truly high number of DOFs, thereby allowing for the appreciation of obtained accuracy and computational speed-ups. We simulate the response of the structure to a low frequency pulse load, applied as a spatially uniform pressure load on the highlighted area on the structure skin (cf. Figure 1). This pressure load takes the shape of a pulse in time as shown in Figure 2. The dynamic load function is given as

$$p(t) = A \sin^2(\omega t) \left[H(t) - H\left(\frac{\pi}{\omega} - t\right)\right], \tag{16}$$

where $H(t)$ is the Heaviside function and ω chosen as the average of the first and second natural frequency of vibration. The load amplitude is chosen so that the linear and nonlinear internal forces have magnitudes of similar order (see Figure 2, cf. [9] for linear and nonlinear response of this model). This is in agreement with the range of applicability of von-Kármán-kinematics-based shell elements, used here.



((a))



((b))

Figure 1: A wing structure with NACA 0012 airfoil (length(L) = 5 m, Width(W) $\approx$ 0.9 m, Height(H) = 0.1 m) stiffened with ribs along the longitudinal and lateral direction. The Young Modulus is $E = 70$ GPa, the Poisson's ratio is $\nu = 0.33$, and the density is $\rho = 2700$ Kg/m^3. The wing is cantilevered at one end. Uniform pressure is applied on the highlighted area, with a pulse load as given by (16) (shown in (a)). The structure is meshed with triangular flat shell elements with 6 DOFs per node and each with a thickness of 1.5 mm. The mesh contains $n = 135770$ DOFs, $n_e = 49968$ elements. For illustration purposes, the skin panels are removed and mesh is shown in (b).

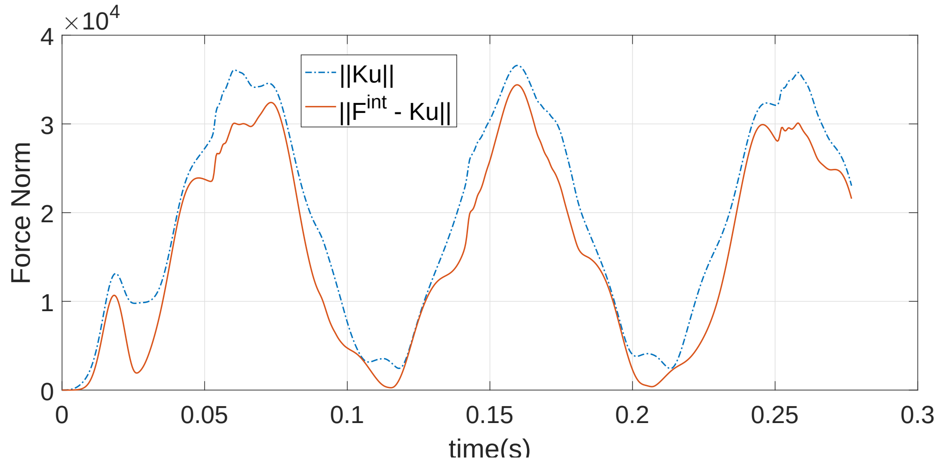


Figure 2: (a) Dynamic load function for pulse loading (cf. (16)). (b) The comparison of the norm of the linear and the nonlinear internal force during a full nonlinear solution of (cf. Figure 1).

The solution to the linearized system can be accurately reproduced using modal superposition with the first five VMs of the structure. These VMs ($M = 5$), along with the corresponding SMDs are used to construct reduction basis to perform simulation-free reduction with $m = 20$, as discussed in Section 5.1. The SIMFREE ROM is further equipped with the quadratic-manifold-based training-set generation technique to perform simulation-free hyper-reduction. using ECSW. The results for these techniques are compared with the classical (hyper-)reduction approaches and reported in Table 1.

Reduction Technique	# el.	$GRE_M(\%)$	$S^\star$	$S^\star_{eff}$
POD	49968	0.31	2.77	0.73
POD-ECSW	**278**	0.38	**394.6**	**0.97**
SIMFREE	49968	1.65	2.72	2.69
SIMFREE-ECSW	**156**	1.40	**638.7**	**38.48**

Table 1: Starting with a linear modal superposition with $M = 5$ modes (first 5 VMs), reduction techniques as described in Section 5.1 are formulated. The size of the reduction basis is $m = 20$ for all the presented cases. Global Relative Error, speed-ups and effective speed-ups (for $c_{on} = c_{off} = 0.5$) for these reduction techniques are tabulated. A total of $n_t = 200$ training vectors, chosen uniformly from the solution time-span, are used to setup hyper-reduction in Modal-ECSW-L1/L2 as well as ECSW-POD. Time for a full solution run $T_{full} = 3.808 \times 10^4$s.

The following observations can be made from the results in Table 1:

- The general need for hyper-reduction is apparent from the results for classical reduction as well as simulation free reduction approaches. Even for a model with truly high number of DOFs the online speed-up ($S^\star$) shows a value between 2 and 3 for these techniques. Indeed, the evaluation and projection of nonlinearity is a major bottleneck for saving computational time.
- For the conservative choice of equal weights assigned to online and offline costs, the Classical reduction methods (POD) show an effective speed-up $S^\star_{eff} < 1$, even when equipped with hyper-reduction (ECSW-POD). This makes such techniques heavily dependent on an expensive database of full solution runs to obtain any reasonable effective speed-up for a range of load cases.

- The simulation-free reduction techniques, on the other hand, lead to effective speed-ups $S^{\star}_{eff} > 1$. Indeed, the reduction basis constructed using modes and SMDs is much cheaper to obtain than a POD basis which requires the HFM solution vectors.
- The simulation-free hyper-reduction using the quadratic-manifold-based training-set generation results in effective speed-ups which are orders of magnitude higher than any other methods. Specifically, the obtained online speed-up is approximately 1.6 times larger than that of ECSW-POD. This is due to the fact that the sNNLS routine selected only 156 elements (in place of 278 for POD-ECSW). The online speed-up $S^{\star}$ for ECSW is inversely proportional to the number of elements selected in sNNLS routine.

6 CONCLUSION

We have introduced a new method to generate training sets for hyper-reduction of geometrically nonlinear structural dynamics problems, without the need of full solution snapshot, thereby reducing offline costs significantly. This method essentially involves the projection of snapshots obtained from the (inexpensive) linear modal superposition solution, on to a quadratic manifold, which is tangent to the corresponding linear modal sub-spaces and captures the second order nonlinear effects. As discussed in Section 4.2, this modal-based techniques results in physically meaningful training vectors, which capture the essential bending-stretching coupling characteristic of geometrically-nonlinear thin-walled structures. The advantage of this method lies in the fact that the linear modal solution is available for any given system at practically no costs.

As remarked earlier, the quadratic-manifold-based training sets still rely on the underlying linear footprint of the model and correct for the relevant nonlinear effects up to the second order. The degree to which this concept can be stretched would be a problem-dependent issue. One can envisage the load cases where this technique is not expected to perform well, at least on long enough times scales. Specifically, for resonant loading of lightly damped systems, the modal amplitudes are expected to grow indefinitely for a linearized system. In such situation, the linear response is not likely to be representative of the nonlinear system. The projection of the modal snapshots on to the quadratic manifold is not expected to return physically meaningful snapshots for the nonlinear system, since the displacements would be constrained from indefinite growth by the inherent nonlinearity.

The computational speed-ups calculated in the numerical results conservatively assume that the online and offline stages of the simulation carry equal costs. However, it should also be noted that, in general, the POD vectors used for training and reduction purposes can be expected to be optimal only for the load case from which these full-solution vectors are initially obtained. Thus, practically, a database of full-solution runs for different load cases would be needed before any meaningful benefits of model-reduction could be observed. We contend the use of simulation-free reduction techniques to ease preliminary geometrically nonlinear analysis of structures, where such an expensive database of full solutions is unavailable or unaffordable. Thus, we think that the naively defined effective speed-ups used here are still indicative.

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