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Application of a Reference Plane Method Calculating Flowfield behind Three-dimensional Elliptic Conical Shock Wave

*SHI Chong-guang, ZHU Cheng-xiang, YOU Yan-cheng**

School of Aerospace Engineering, Xiamen University, Xiamen 361005, China
Chongguang.Shi@outlook.com

Abstract

In order to solve the three-dimensional flowfield behind the elliptic conical shock wave, a reference plane method (RPM) is applied at the cylindrical coordinate system. The main idea is to transform the Cartesian coordinate system into the cylindrical coordinate system. Then the three-dimensional formulas can be easily applied to calculate the projection flowfield at the reference plane. When the whole projection flowfield has been solved at the cylindrical coordinate system, the actual three-dimensional flowfield at the Cartesian coordinate system will be acquired by the utility of geometry conversion. To validate the applicability of the reference plane method, first it's applied to the concave circle conical shock wave and compare with the MOC's result. It shows that the flowfields solving by the reference plane method and the method of characteristic are the same, which demonstrates the applicability of the reference plane method to solve the axisymmetric flowfield. Then the reference plane method is utilized to elliptic conical shock wave for a relatively large range of attack angle. The shock wave shape and the flowfield behind the elliptic conical shock wave at zero and three degrees angle of attack calculated by the reference plane method are in good agreement with the CFD's. Furthermore, the error of static pressure ratio is 0.8% and 0.6% respectively, which means the reference plane method is also capable for solving the three-dimensional flowfield behind the elliptic conical shock wave with attack angle.

Keywords: *Three-dimensional shock wave; Coordinate conversion; Supersonic flowfield; Reference plane Method.*

Nomenclature

x, r, Φ – Cylindrical coordinates
a – sonic speed
p – static pressure
q – velocity in the reference plane
u – velocity component in flow direction in the reference plane
v – velocity component normal to flow direction in the reference plane
w – velocity component normal to the reference plane
ρ – density
θ – flow angle
μ – Mach angle
γ – Ratio of specific heats
Subscripts
0 – along the streamline
± – along the characteristic

1. Introduction

The most distinctive feature of a supersonic flow is shock waves [1]. Therefore, how to solve the flowfield behind the shock wave is one of the key factors to the improvement of the vehicle performance. With the advantages of fast calculating speed and high-resolution shock wave, method of Characteristics is wildly used in calculating the post-shock flowfield. Sobieczky introduced a design method for hypersonic waverider with given shock wave based on the method of characteristics [2].

Eggers designed advanced waveriders with high aerodynamic efficiency by use of the method of characteristics [3]. Sobieczky generated the aerospace vehicle configurations with supersonic leading edge as well as inlet diffusers by introducing an inverse method of characteristics into the design concept of using osculating cones in the supersonic flow [4]. Yancheng You designed the internal waverider-derived inlet on the basis of the internal compression flowfield that solved by two-dimensional method of characteristics [5]. Barkmeyer presented a design method of an integrated forebody-inlet waverider structure with an isentropic compression inlet in inviscid flow [6]. Matthews use the method of characteristics to acquire internally compressive, axisymmetric flowfields with either constant slope or constant pressure boundary conditions and then built a four-module waverider intake model [7]. However, the two-dimensional method of characteristics is generally applied at the Cartesian coordinate system, which restricts the range of usability. Therefore, some other methods are developed by applying to different coordinate system. Sirovich introduced a coordinate system consisting of the principal characteristics and streamlines and then develop an approximate solution for two-dimensional, steady, inviscid supersonic flow over an airfoil [8-10]. Shi Chongguang presented a series of algebraic methods to solve the two-dimensional planar flowfield behind the curved shock at the streamline-characteristic coordinate system [11]. Mölder derived the curved shock equations including terms reflecting upstream vorticity, upstream flow non-uniformity and compound shock curvature at a coordinate system parallel and normal to the streamlines [12]. Although algebraically cumbersome, the equations can be applied in subsonic flow regions, which are more versatile than the method of characteristics. However, all the methods above are suitable for two-dimensional planar and axis-symmetry flow. In three-dimensional flow, the streamlines deflect because of the lateral velocity, which makes the flow much more complicated. Rakich described a reference plane method of characteristics and applied the method to inclined bodies of revolution [13-14]. With the technology development, the vehicles are required to work reliably under extreme conditions. the basic flowfield based on bodies of revolution are unable to meet all requirements.

As is known, when the cross-sectional area of an elliptic cone per unit length is the same as a circular cone, the elliptic cone has significantly higher lift-drag ratio. With many other advantages, studies of the elliptic conical flowfield are of interest [15-17]. As the calculating ability of computers increased rapidly, the flow past an elliptic cone can be easily solved utility of CFD. However, sometimes the reverse problem with given shock wave is badly in need. In order to solve the three-dimensional flowfield behind the elliptic conical shock wave, a reference plane method (RPM) is applied at the cylindrical coordinate system. The main idea is to calculate the projection flowfield at the reference plane. The method is applied to elliptic conical shock wave for a relatively large range of attack angle to validate the applicability.

2. The reference plane method

2.1. Equations of the reference plane method

The equations governing the steady, three-dimensional flow of an inviscid gas at the Cylindrical coordinates system are as follows:

$$u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial r}+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial r}=-\frac{w}{r}\frac{\partial \rho}{\partial \phi}-\frac{\rho}{r}\frac{\partial w}{\partial \phi}-\frac{\rho v}{r} \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{w}{r}\frac{\partial u}{\partial \phi}-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}=-\frac{w}{r}\frac{\partial v}{\partial \phi}-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{w^2}{r} \tag{3}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}=-\frac{w}{r}\frac{\partial w}{\partial \phi}-\frac{1}{\rho r}\frac{\partial p}{\partial \phi}-\frac{vw}{r} \tag{4}$$

$$u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial r}-a^2\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial r}\right)=-\frac{w}{r}\frac{\partial p}{\partial \phi}+a^2\frac{w}{r}\frac{\partial \rho}{\partial \phi} \tag{5}$$

To simplify the calculation process, the reference plane method is introduced with the idea that the three-dimensional flowfield can be projected onto many reference planes. Then like the method of characteristics applied to axis-symmetry conditions, the scheme is processed on the reference planes. To solve the flowfield on the reference planes, the characteristic equations and compatibility equations are derived from Eq.1-5. With the algebraic manipulation, the streamline and characteristic directions are given by the equations:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)_0=\lambda_0=\frac{v}{u} \tag{6}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)_\pm=\lambda_\pm=\tan\left(\theta\pm\mu\right) \tag{7}$$

Along the characteristic curves and streamlines, the compatibility relation takes the form as follows:

$$\mathrm{d}_\pm\theta\pm\frac{\sqrt{q^2-a^2}}{\rho a q^2}\mathrm{d}_\pm p=-\frac{F_1\cos\mu\pm F_2\sin\mu}{\cos\left(\theta\pm\mu\right)}\mathrm{d}_\pm x \tag{8}$$

$$\rho q\mathrm{d}_0 q+\mathrm{d}_0 p=\left(-\frac{w}{r}\frac{\partial q}{\partial \phi}+\frac{w^2}{r}\sin\theta\right)\rho\mathrm{d}_0 s \tag{9}$$

$$q\mathrm{d}_0 w=\left(-\frac{w}{r}\frac{\partial w}{\partial \phi}-\frac{1}{\rho r}\frac{\partial p}{\partial \phi}-\frac{vw}{r}\right)\mathrm{d}_0 s \tag{10}$$

$$q\left(\mathrm{d}_0 p-a^2\mathrm{d}_0\rho\right)=\left(-\frac{w}{r}\frac{\partial p}{\partial \phi}+a^2\frac{w}{r}\frac{\partial \rho}{\partial \phi}\right)\mathrm{d}_0 s \tag{11}$$

Where,

$$F_1=\frac{w^2}{q^2 r}\cos\theta-\frac{w}{qr}\frac{\partial \theta}{\partial \phi} \tag{12}$$

$$F_2=\frac{w}{\gamma qpr}\frac{\partial p}{\partial \phi}-\frac{w}{q^2 r}\frac{\partial q}{\partial \phi}+\frac{1}{qr}\frac{\partial w}{\partial \phi}+\frac{\sin\theta}{r}\left(1+\frac{w^2}{q^2}\right) \tag{13}$$

It is noteworthy that with no lateral velocity, the flow should be symmetrical. And from the characteristic equations and compatibility equations, it's clear that when the lateral velocity is zero, the equations are exactly simplified to two-dimensional method of characteristics. The correspondence validates the downward compatibility of the reference method.

2.2. Scheme of the reference plane method

The difference between two-dimensional and three-dimensional flow is the lateral velocity. Owing to the existence of the lateral velocity, the three-dimensional compatibility equations are more complicated than the two-dimensional equations, which means there should be more iterations in the scheme of the reference plane method. The iterations can be broadly divided into two categories. First, the scheme on the reference plane is similar to the process of two-dimensional method of characteristics. After iterations between three characteristic curves, the parameters on the reference planes are acquired. Then the second part is the scheme between the reference planes. With the iterations between the reference planes, the lateral velocity is solved. The specific solution procedure is presented in the Fig.1. And the illustration is as follow:

1) Calculate the post-shock parameters based on the oncoming flow and the shock wave

2) On the basis of the assumption that the lateral velocity is zero, solve the parameters along the characteristic direction
3) Solve the lateral velocity with the acquired parameters
4) Update the parameters along the characteristic direction with the lateral velocity
5) Iterate over the step 2 and 3 until the residual is less than the set value
6) Repeat the step 2, 3 and 4 to solve the flowfield on all the reference planes
7) Acquire the actual physical post-shock flowfield after the geometric transformation

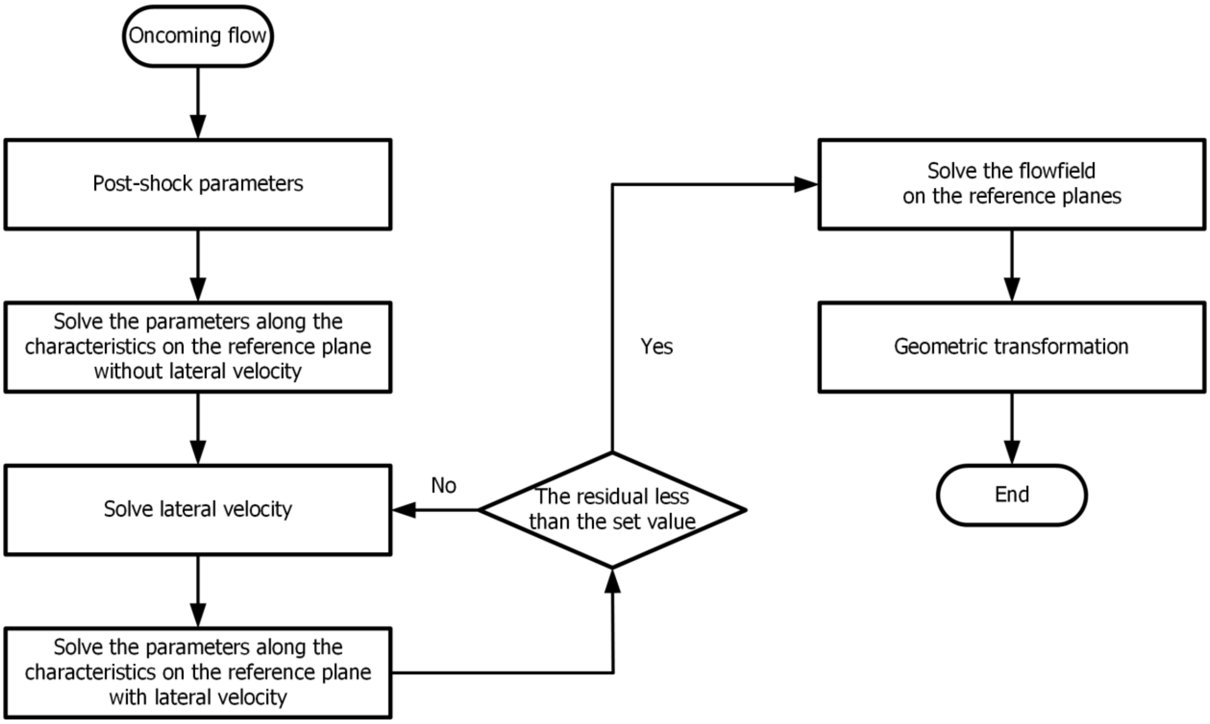


Fig 1. Flow chart of the reference method

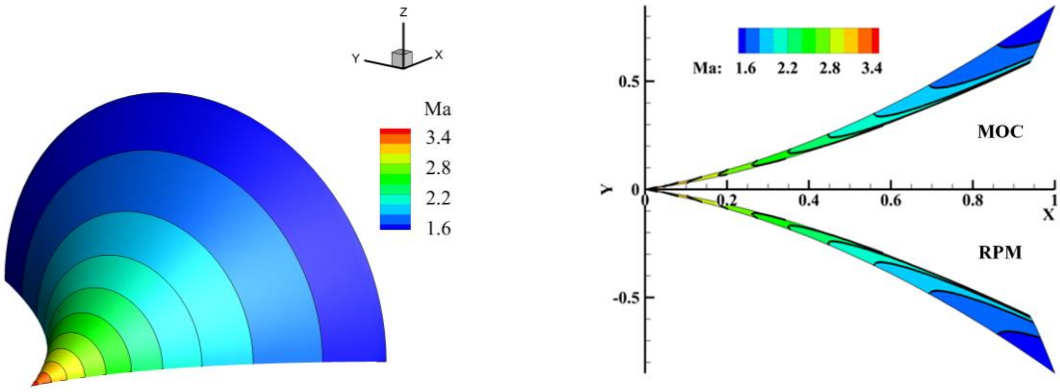
Although there are many iterations in the scheme, the convergence rate is fast. In general, the CPU time to solve a mesh size of 70*70*120 is no more than 150 seconds.

3. Application of the reference plane method

To validate the applicability of the reference plane method, it's applied to the two-dimensional axis-symmetry flow first and then three-dimensional flow with given shock. Since the inverse design method of characteristics is able to calculate the post-shock flowfield in two-dimensional flow, the results are used as a comparison. As for three-dimensional flow, the flowfields solved by the reference plane method are compared with CFD's.

3.1. Concave conical shock wave

To verify downward compatibility of the reference plane method, first it's used to solve the flowfield behind the concave circle conical shock wave. The equation of concave conic generatrix is $y=0.5x^2+0.35x$ and the oncoming flow Mach number is 4. Fig.2 is the post-shock flowfield solved by method of characteristics (MOC) and reference plane method (RPM). Since the circumferential pressure maintains unchanged, the three-dimensional flowfield is simplified to an axisymmetric one, which is showed in the Fig.2(a). Extract the flowfield in the reference plane and compare with the MOC's result. The MOC's and RPM's results are in the first and fourth quadrant respectively. From Fig.2(b), it is clear that the flowfields solving by the reference plane method and the method of characteristic are the same, which demonstrates the downward compatibility of the reference plane method to solve the axisymmetric flowfield. It is a foregone conclusion since when the lateral velocity is zero, the equations mentioned in section 2.1 are exactly simplified to two-dimensional method of characteristics.



(a) three-dimensional flowfield (b) symmetry flowfield

Fig 2. Comparison between MOC and RPM results of the concave conical shock wave

3.2. Elliptic conical shock wave without/with angle of attack

Then the reference plane method is utilized to elliptic conical shock wave for a relatively large range of attack angle. The first case is a standard elliptic cone and the ratio of major axis to minor axis is 1.2. The oncoming flow Mach number is 4. Due to the nonaxisymmetry, the lateral pressure gradient exists, which makes the flow deflect. Thus, the initial flowfield is the projection of the three-dimensional physical flowfield. To acquire the actual flowfield, the coordinate system conversion is needed. Since there is no theoretical solution of the elliptic conical flowfield, the CFD results have been acquired to demonstrate the accuracy of the reference plane method. The CFD mesh is shown in the Fig.3. The quantity of the mesh is about 3.73 million. The x-axis is parallel to the direction of the oncoming flow.

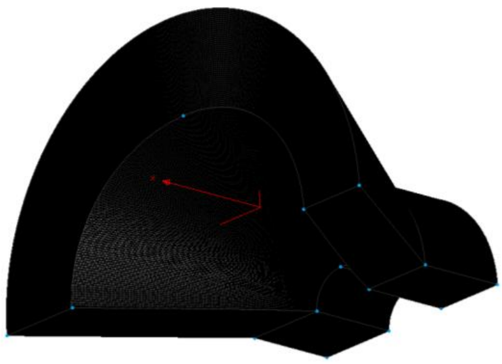
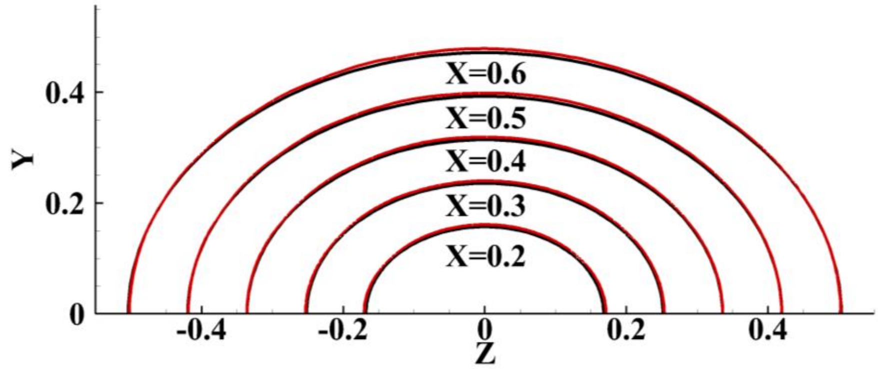
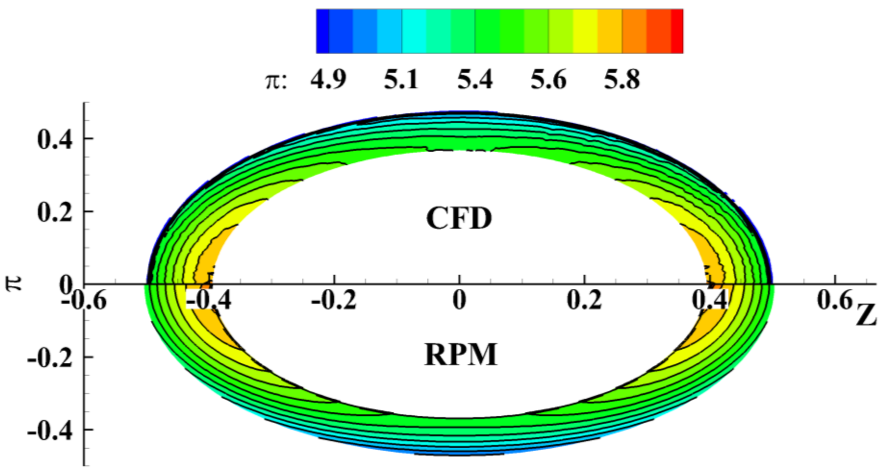


Fig 3. CFD mesh

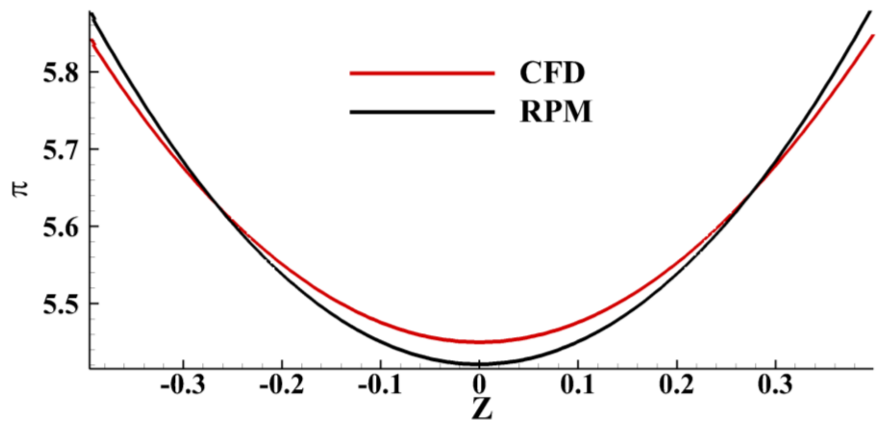
The inviscid CFD result of the elliptic cone without angle of attack is shown in Fig.4, in comparison with the flowfield solved by reference plane method. The comparisons of the shock wave shape, the flowfield and the pressure ratio are in the Fig.4(a), (b), (c) respectively. In terms of the shock wave, the curves are extracted at x=0.2, 0.3, 0.4, 0.5 and 0.6. The red curves stand for the CFD results and the black curves represent the RPM results. From the picture, it's known that the shock wave shapes calculated by the two methods are almost the same. In terms of the flowfield, the slices at x=0.6 are extracted and shown in the Fig.4(b). The CFD's result is in the first quadrant and the RPM's result is in the fourth quadrant. From the picture, it's known that the pressure distribution of the RPM is similar to the CFD's, except in the close proximity of the shock wave. That's because the shock wave solved by CFD is a layer with thickness, not a slice like the theoretical method, which is one of the advantages of the theoretical method. As for wall pressure distribution in Fig.4(c), the red and the black curves stand for the CFD and the RPM results respectively, same as the Fig.4(a). The picture shows that the RMP's wall pressure is higher than the CFD's on both sides but lower in the middle. In general, there is a good coincidence that the error of static pressure ratio at incoming Mach number 4 is 0.8%, which validates the reference plane method is capable to solve the flowfield behind the elliptic conical shock wave without angle of attack.



(a) shock wave



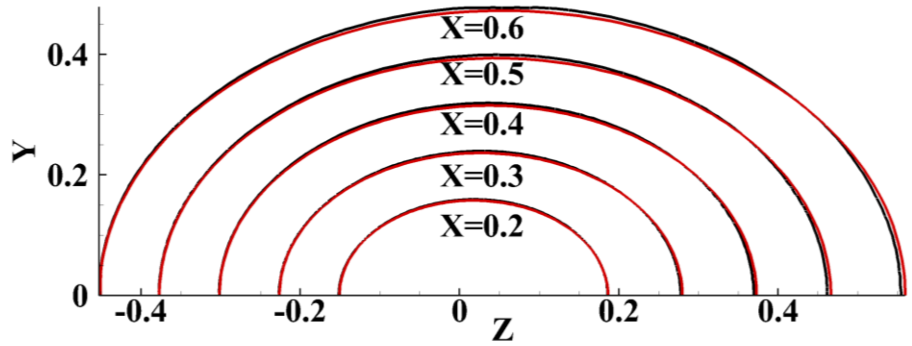
(b) distribution of pressure ratio (X=0.6)



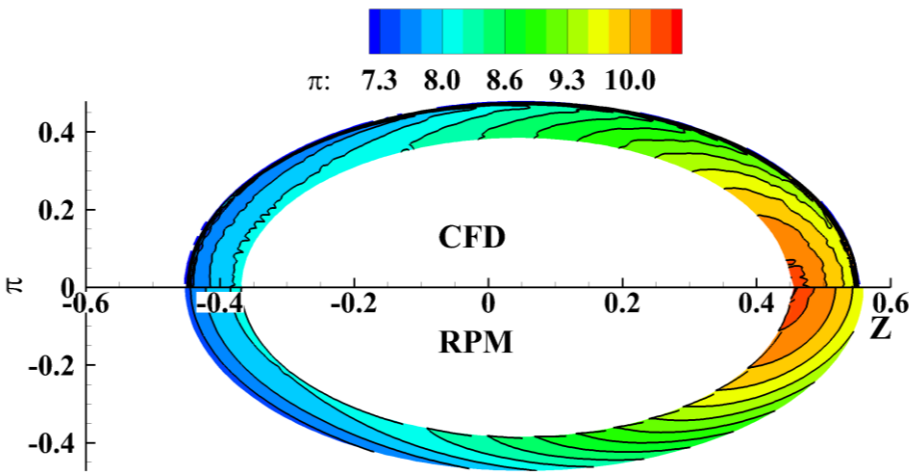
(c) wall pressure distribution (X=0.6)

Fig 4. Comparison between CFD and RPM results of the elliptic conical shock wave without attack angle

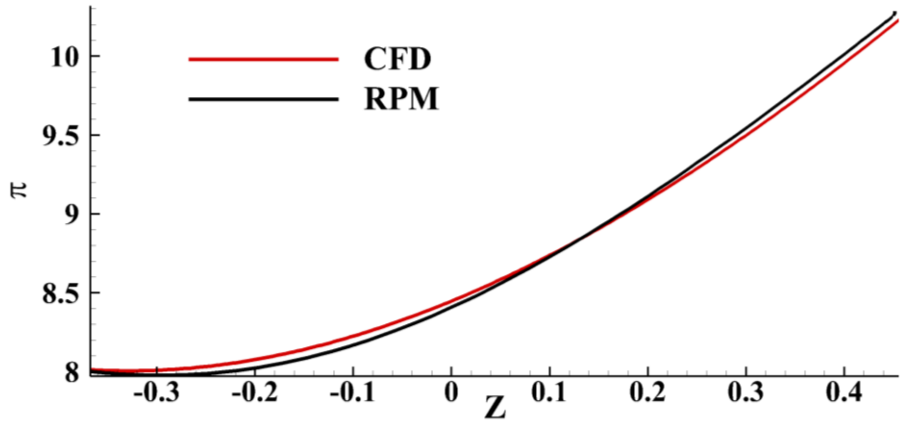
The second case is an elliptic cone same as the first one but with three degrees angle of attack. The ratio of major axis to minor axis is 1.2 and the oncoming flow is parallel to the x-axis with the Mach number 5. The CFD mesh distribution is same as above case. The quantity of the mesh is about 3.73 million. Fig.5 presents the inviscid CFD result of the elliptic cone with three degrees angle of attack, in comparison with the flowfield solved by reference plane method. Fig.5(a), (b), (c) are the pictures of comparisons of the shock wave shape, the flowfield and the pressure ratio respectively. In Fig.5(a), the red curves stand for the CFD results and the black curves represent the RPM results. The shock wave curves are extracted at different position along the x-axis. From the picture, it's clear that the shock wave shapes calculated by the two methods are almost the same. There are minute differences on both sides and in the middle. In Fig.5(b), the slices of the flowfield at x=0.6 are extracted and compared. The CFD's result is in the first quadrant and the RPM's result is in the fourth quadrant. From the picture, it's known that the flowfield solved by the RPM has a similar pressure distribution with the CFD's, except in the close proximity of the shock wave. The shock wave of CFD has a thickness owing to the mesh size whereas the shock wave solved by RPM is a slice, which is one of the advantages of the reference plane method. As for wall pressure distribution in Fig.5(c), the red and the black curves stand for the CFD and the RPM results respectively, same as usual. The picture shows that the RMP's wall pressure has a good coincidence with the CFD's in the middle but small discrepancies on both sides. The error of static pressure ratio at incoming Mach number 5 is about 0.6%, which means the reference plane method also has the ability to solve the three-dimensional flowfield behind the elliptic conical shock wave with attack angle.



(a) shock wave



(b) distribution of pressure ratio (X=0.6)



(c) wall pressure distribution (X=0.6)

Fig 5. Comparison between CFD and RPM results of the elliptic conical shock wave at three degrees angle of attack

Fig.6 is a picture of the relative error of static pressure. The given shock wave shape is an elliptic cone with three degrees angle of attack. The ratio of major axis to minor axis increases to 1.5. The oncoming Mach number varies from 3 to 10. The CFD mesh distribution is same as above. The quantity of the mesh is about 3.73 million. From the picture, it's known that between Mach number 5 and 10, the error of the static pressure remains steady with about 2.5%. However, When the oncoming flow Mach number is less than 5, the error of static pressure increases. The error is 4.11% at oncoming Mach number 3 and 3.30% at oncoming Mach number 4. The reason is that at the lower Mach number, the pressure ratio through the shock wave has a low value so that the error would amplify with the transformation from absolute error to relevant error. Generally, the relevant errors of static pressure are less than 5%, and with the increase of oncoming Mach number, the relevant error declines and then maintains at a low level after Mach number 5. Therefore, with all these validation examples, it is clear that the reference plane method is qualified to be as a design tool for the elliptical basic flowfield, which can be used to design the vehicle afterwards.

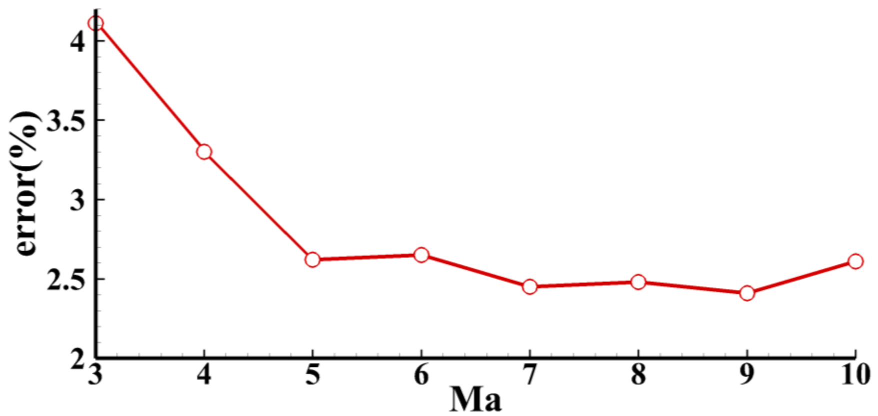


Fig 6. The relative error of static pressure at different oncoming Mach number

4. Conclusion

The reference method can be applied to solve the post-shock flowfield not only in two-dimensional flow but also in three-dimensional elliptic conical shock wave. The reference plane method is utilized to calculate the post-shock flowfield in axis-symmetry flow and the result coincides with the MOC's. Furthermore, the flowfields behind the elliptic conical shock wave for a relatively large range of attack angle have been solved by the reference plane method. The shock wave shape and the post-shock flowfield at zero and three degrees angle of attack calculated by the reference plane method are in good agreement with the CFD's. The error of static pressure ratio is 0.8% and 0.6% respectively. Therefore, the reference plane method can be used to design a vehicle by solving three-dimensional flowfield behind the elliptic conical shock wave without/with attack angle as the basic flowfield.

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